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Heterotic SO(32) model building in four dimensions

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Abstract: Four dimensional heterotic SO(32) orbifold models are classified systematically with model building applications in mind. We obtain all $\mathbb{Z}_3$, $\mathbb{Z}_7$ and $\mathbb{Z}_{2N}$ models based on vectorial gauge shifts. The resulting gauge groups are reminiscent of those of type-I model building, as they always take the form $\mathrm{SO}(2n_0) \times \mathrm{U}(n_1) \times \cdots \times \mathrm{U}(n_{N-1}) \times \mathrm{SO}(2n_N)$. The complete twisted spectrum is determined simultaneously for all orbifold models in a parametric way depending on $\mathrm{n}_0, \ldots, \mathrm{n}_\mathrm{N}$, rather than on a model by model basis. This reveals interesting patterns in the twisted states: They are always built out of vectors and anti-symmetric tensors of the $\mathrm{U}(n)$ groups, and either vectors or spinors of the $\mathrm{SO}(2n)$ groups. Our results may shed additional light on the S-duality between heterotic and type-I strings in four dimensions. As a spin-off we obtain an SO(10) GUT model with four generations from the $\mathbb{Z}_4$ orbifold.

Keywords: Superstrings and Heterotic Strings, String Duality, Superstring Vacua.

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1. Introduction and summary

Since the mid eighties there have been many studies to the physics of extra dimensions. This route was first considered seriously with the development of superstrings, and in particular when it was realized that the heterotic $\mathrm{E}_8 \times \mathrm{E}_8$ string [1, 2] can give rise to four dimensional phenomenology [3] by considering Calabi-Yau or orbifold compactification [4, 5]. Conformal field theories on the latter spaces [6] are particularly simple since they are free. The extension to orbifolds with gauge field background, or Wilson lines, has been first investigated in ref. [7]. A major part of the literature has been devoted to the heterotic $E_8\times E_8$ theory, that was the first string theory to be considered seriously for Standard Model (SM) phenomenology. This was due mainly to the fact that even for the simplest standard embedding of the spin connection in the gauge group, Grand Unified Theory (GUT) gauge groups arise from one of the E_8 gauge groups, while the other E_8 group can be considered as part of a hidden sector that might be responsible for supersymmetry breaking by gaugino condensation [8, 9, 10]. After these developments there have been many efforts to obtain a full picture of all possible gauge groups that can arise from heterotic $E_8 \times E_8$ orbifolds [11]–[16]. For recent investigations to heterotic string Supersymmetric Standard Model (MSSM) and GUT model building we refer to [17]–[20] and references therein.

The study of string phenomenology turned a different direction with the construction of D-branes [21, 22] in type-II string theory. A stack of D-branes gives rise to $U(n)$ or $SO(2n)$ gauge groups, and therefore models with various stacks of branes lead to effective theories with products of such gauge groups. The cancellation of RR-flux tadpoles selects consistent D-brane models [23]. In particular, the type-I string can be viewed as a type-II orientifold. By considering branes at angles [24, 25] it has been possible to construct orientifold models with similar gauge group and spectrum as the Standard Model or its supersymmetric extension [26]–[29]. An interesting aspect is that type-I string and the heterotic SO(32) string are related via a strong/weak duality in ten dimensions [30]. Therefore also the phenomenology of the heterotic SO(32) orbifold models should be studied in detail. Some steps in that direction have been taken refs. [31, 32], where $\mathbb{Z}_3$ orbifolds with Wilson lines were considered. And [33] investigates discrete symmetries like CPT in the (SO(32)) string theory. We seek to obtain a full classification of more general orbifolds in the heterotic SO(32) string context, but for the sake of simplicity we ignore the possibility of Wilson lines. This investigation may shed additional light on the S-duality between heterotic and type-I strings in four dimensions.

Another motivation for our pursuit of full classification of heterotic SO(32) models is that they may be useful extension of field theory models of extra dimensions. In recent years there has been a lot of interests in five, six and higher dimensional orbifold field theories [34]–[39] and orbifold GUTs [40]–[43], making use of split multiplets for the Higgs [44]. Essentially all these models are non-renormalizable and therefore require some form of ultra-violet completion. At the moment the only candidates for complete theories in extra dimensions come from string theory. A concrete example for some orbifold GUTs have been obtained from heterotic string theory, by taking a-symmetric limits [45] of some of the radii of the $\mathbb{Z}_6$ orbifold [46, 47], for a general investigation to the scales in such a scenario see also [48]. Many of the heterotic SO(32) models, that we classify in this work, may be used for field theoretical investigations in a similar way.

The main results of our work can be summarized as: We give a systematic classification of four dimensional heterotic SO(32) orbifold models. We obtain all $\mathbb{Z}_3$, $\mathbb{Z}_7$ and $\mathbb{Z}_{2N}$ models based on vectorial gauge shifts. The resulting gauge groups are reminiscent of those obtained in type-I model building as they generically take the form: $SO(2n_0) \times U(n_1) \times \cdots \times U(n_{N-1}) \times SO(2n_N)$. Most classification works for $E_8 \times E_8$ orbifolds stop here, once the resulting gauge group has been obtained. We continue to determine the complete twisted spectrum simultaneously for all orbifold models, rather than on a model by model basis. This reveals interesting patterns in the twisted states that are manifestly portrayed in our classification tables. For example we give explicit mappings between various twisted spectra, which greatly reduces the classification effort. The paper is outlined as follows.

In section 2 and 3 we review the classification of $\mathbb{Z}_3$ and $\mathbb{Z}_4$ orbifold models respectively, as simple examples of odd and even order orbifold models and to fix the notation. We also compute the anomaly polynomial in order to have a consistency check on the spectrum. The anomaly analysis allows us to understand how the only heterotic model having a type-I counterpart in the $\mathbb{Z}_3$ case is free from irreducible anomalies already at the level of untwisted states, while the other models, with one exception, do not have this feature. Interestingly, the exception has gauge group equal to the original SO(32), which, to our knowledge, has no type-I dual.

Section 4, the core of our paper, explains the details of our classification procedure. We first reduce the problem of finding modular invariant shifts to an exercise in linear algebra. In order to perform the classification, we restrict ourselves to those twisted states, that cannot be obtained from other twisted states by orbifolding. By exploiting spectral flow we bring all weights in a standard form so that classification becomes very simple. In this way we show that the structure of the twisted states is always the same. We recompute the twisted $\mathbb{Z}_3$ and $\mathbb{Z}_4$ to illustrate how efficient our classification procedure is. We complete the classification prescription by returning to the reducible twisted states and explain how to compute them from the irreducible ones using primarily field theory methods.

In section 5 we give various possible applications and extensions of our results. We first explain classification procedure can be easily extended to $\mathbb{Z}_N \times \mathbb{Z}_{N'}$ orbifold models and to $\mathrm{E}_8 \times \mathrm{E}_8$ heterotic string. Next, we consider the heterotic/type-I S-duality in the light of our classification on the heterotic side. For each of the odd order orbifolds ($\mathbb{Z}_3$ and $\mathbb{Z}_7$) there are only two models free of irreducible anomalies at the untwisted spectrum level. But only one has a type-I counterpart, while the other, with gauge group SO(32), does not have a type-I dual.

In appendix A we describe the modular invariant partition function of a generic $\mathbb{Z}_N$ orbifold model that is basis of our classification. In the other appendices we give the classification of $\mathbb{Z}_6$, $\mathbb{Z}_7$ and $\mathbb{Z}_8$ models. There one finds a classification of the modular invariant shifts and tables of the irreducible spectra. For $\mathbb{Z}_7$ model we also give the anomaly polynomial corresponding to the irreducible anomaly.

1.1 Technical preliminaries

We consider the heterotic SO(32) theory on a six dimensional torus T^6. (We will ignore the possibility of having various radii for these tori for simplicity.) To obtain a chiral spectrum this torus must be orbifolded. To this end we write $T^6 = T^2 \times T^2 \times T^2$ parameterized by the complex coordinates z^i, $i = 1, 2, 3$. A compact orbifold is obtained by the identification

$$z^i \to e^{2\pi i\, \phi_i} z^i \,, \tag{1.1}$$

due to a discrete abelian group that we take to be $\mathbb{Z}_N$. Crystallography of the torus lattices only allow $N = 2, 3, 4, 6, 7, 8$ and 12. Preservation of supersymmetry leads to the requirement that

$$\sum_i \phi_i \equiv 0\,, \tag{1.2}$$

where the equivalence relation $a \equiv b$ indicates that a and b differ by an integer. All the standard shifts are chosen such that equality holds

$$\begin{aligned} &\phi_{\mathbb{Z}_3} = \frac{1}{3}(1,1,-2)\,, \qquad \phi_{\mathbb{Z}_4} = \frac{1}{4}(1,1,-2)\,, \qquad \phi_{\mathbb{Z}_{6\text{-I}}} = \frac{1}{6}(1,1,-2)\,, \\ &\phi_{\mathbb{Z}_7} = \frac{1}{7}(1,2,-3)\,, \qquad \phi_{\mathbb{Z}_{8\text{-I}}} = \frac{1}{8}(1,2,-3)\,. \end{aligned} \tag{1.3}$$

In string the spacetime coordinate become bosonic fields X^M on the string worldsheet. As the heterotic theory only contains closed strings, the ten coordinate field X^M can be expanded

$$X^M(\sigma_0,\sigma_1) = \sum_k \left(e^{2\pi i\, k(\sigma_1-\sigma_0)} \alpha_k^M + e^{2\pi i\, k(\sigma_1+\sigma_0)} \tilde{\alpha}_k^M \right) \tag{1.4}$$

in both left- and right-moving oscillators, α_k^M and $\tilde{\alpha}_k^M$, respectively. Excited states are created from the vacuum $|0\rangle$ by acting on it with these oscillators. To facilitate the identification with field theoretical compactification it is useful to define such states with lower indices. For example,

$$|{}_M\rangle = \eta_{MN} \alpha_{-k}^N |0\rangle\,, \tag{1.5}$$

gives a target space vector. This is compatible with the fact that gauge fields being connections have their spacetime index downstairs: A_M. On the right-moving side the worldsheet theory is supersymmetric, and contains ten real fermions $\tilde{\psi}^M$. Instead, on the left-moving side the theory can be described as having 16 complex fermions λ^I and hence this side is not supersymmetric. Here I labels 16 Cartan generators of SO(32). The orbifold action is embedded in the λ^I through a 16 dimensional vector $v = (v^I)$

$$\lambda^I \to e^{2\pi i\, v_I} \lambda^I\,. \tag{1.6}$$

This complete our technical introduction.

2. $\mathbb{Z}_3$ models

We investigate the heterotic SO(32) theory compactified on the six dimensional orbifold $T^6/\mathbb{Z}_3$ with space action $\phi = \frac{1}{3}(1,1,-2)$. The orbifold action on the gauge group is defined by the shift vector

$$v = \frac{1}{3}\left(0^{16-3n}, 1^{2n}, -2^n\right)\,, \qquad 0 \le n \le 5\,, \tag{2.1}$$

satisfying the modular invariance constraint $\frac{3}{2}v^2 \equiv 0$. For $n > 1$ this is a multiple embedding of the so-called standard embedding with $n = 1$. (In the $E_8 \times E_8$ such multiple embeddings can always be reduced to simpler embeddings. This is not the case in the SO(32) theory because, contrary to the E_8 case, there are no spinorial roots in which one can perform Weyl reflections.) At the fixed points the SO(32) gauge symmetry is broken to

$$\mathrm{SO}(32) \to \mathrm{SO}(32-6n) \times \mathrm{U}(3n)\,. \tag{2.2}$$

The untwisted matter can determined by considering the branching of the adjoint of SO(32):

$$\begin{aligned}\mathbf{496} \to & \left(\frac{\mathbf{1}}{\mathbf{2}}(\mathbf{32}-\mathbf{6n})(\mathbf{31}-\mathbf{6n}),\mathbf{1}\right)_0 + (\mathbf{1},(\mathbf{3n})^{\mathbf{2}})_0 + \left(\mathbf{1},\frac{\mathbf{1}}{\mathbf{2}}\mathbf{3n}(\mathbf{3n}-\mathbf{1})\right)_2 + \\ & + \left(\mathbf{1},\overline{\frac{\mathbf{1}}{\mathbf{2}}\mathbf{3n}(\mathbf{3n}-\mathbf{1})}\right)_{-2} + (\mathbf{32}-\mathbf{6n},\mathbf{3n})_1 + (\mathbf{32}-\mathbf{6n},\overline{\mathbf{3n}})_{-1}\,.\end{aligned} \tag{2.3}$$

The U(1) charge operator $q = (0^{16-3n}, 1^{3n})$ is fixed by the requirement that the SO($32-6n$) roots $(\underline{\pm 1^2, 0^{14-3n}}, 0^{3n})$ and the SU($3n$) roots $(0^{16-3n}, \underline{1, -1, 0^{3n-2}})$ are neutral. The underline denotes all possible permutations. In the case $n=5$ there are no SO roots but instead there is an additional charge operator $q' = (1, 0^{15})$. Of course for $n=0$ the gauge group is SO(32). The SO(32) roots representation of the untwisted matter reads

$$v\cdot w - \frac{1}{3} \equiv 0: \quad \begin{cases} (\mathbf{32}-\mathbf{6n},\mathbf{3n})_1 : w = (\underline{\pm 1, 0^{15-3n}}, \underline{1, 0^{3n-1}})\,, \\ (\mathbf{1},\frac{\mathbf{1}}{\mathbf{2}}\mathbf{3n}(\mathbf{3n}-\mathbf{1}))_{-2} : w = (0^{16-3n}, \underline{-1^2, 0^{3n-2}})\,. \end{cases} \tag{2.4}$$

These untwisted states form a triplet under the $\mathrm{SU}(3)_H$ holonomy group; this leads to a multiplicity of three for the untwisted spectrum. This untwisted matter results in the anomaly polynomial

$$\begin{aligned} I_{6|u} = & -\frac{1}{6}27(4-n)\mathrm{tr}F_{\mathrm{SU}}^3 + \frac{1}{6}27\cdot 6n\cdot(n-2)F^3 - \frac{1}{48}27n(3n-11)F\mathrm{tr}R^2 + \\ & + \frac{1}{2}36(n-3)F\mathrm{tr}F_{\mathrm{SU}}^2 - \frac{1}{2}9nF\mathrm{tr}F_{\mathrm{SO}}^2\,, \end{aligned} \tag{2.5}$$

that encodes the structure of the pure and mixed anomalies. (We recall that for an anti-symmetric tensor representation $[\mathbf{r}]_2$ of any representation $\mathbf{r}$ of dimension $|\mathbf{r}|$ the trace identities hold: $\mathrm{tr}_{\mathbf{r}_2}F^2 = (|r|-2)\mathrm{tr}_{\mathbf{r}}F^2$ and $\mathrm{tr}_{\mathbf{r}_2}F^3 = (|r|-4)\mathrm{tr}_{\mathbf{r}}F^3$.) Here $F_{\mathrm{SO}}, F_{\mathrm{SU}}$ and F are the field strength two-forms of the SO($32-6n$), SU($3n$) and U(1) gauge symmetries, respectively, and R denotes the curvature two-form. The traces of the SO($32-6n$) and SU($3n$) field strengths are evaluated in the vector representation unless otherwise indicated. In the $n=5$ model also the field strength F' is present in the anomaly polynomial. As the pure U(1)$'$ anomaly and all mixed anomalies involving a single F' vanish immediately, it only gives a single extra term $-45\,F\,F'^2$ replacing the contribution $-\frac{1}{2}9n\,F\,\mathrm{tr}\,F_{\mathrm{SO}}^2$.

The twisted states either are singlets or triplets of the $\mathrm{SU}(3)_H$ holonomy group. They are determined by the relations

$$\mathbf{1}_H:\quad \frac{1}{2}(w+\tilde{v})^2 = \frac{2}{3}\,, \qquad \overline{\mathbf{3}}_H:\quad \frac{1}{2}(w+\tilde{v})^2 = \frac{1}{3}\,, \qquad \mathbf{3}_H + \overline{\mathbf{6}}_H:\quad \frac{1}{2}(w+\tilde{v})^2 = 0\,, \tag{2.6}$$

where $\tilde{v} = (0^{16-3n}, \frac{1}{3}^{3n})$. Here the $\overline{\mathbf{6}}_H$ representation arises as the symmetric two-index tensor representation. This and the representation $\mathbf{3}_H$ can only appear in the full SO(32) orbifold, i.e. the case $n=0$.

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n	$\mathbf{1}_H$	weights	$\overline{\mathbf{3}}_H$	weights	$\mathbf{3}_H+\overline{\mathbf{6}}_H$	weights
0					$\mathbf{1}$	(0^{16})
1	$(\mathbf{1},\mathbf{1})$, $(\mathbf{26},\mathbf{1})$	$(0^{13},-1^3)$, $(\underline{\pm1,0^{12}},0^3)$	$(\mathbf{1},\overline{\mathbf{3}})$	$(0^{13},\underline{-1,0^2})$		
2	$(1,\overline{\mathbf{15}})$	$(0^{10},\underline{-1^2,0^4})$	$(\mathbf{1},\mathbf{1})$	(0^{16})		
3	$(\mathbf{1},\overline{\mathbf{9}})$	$(0^7,\underline{-1,0^8})$				
4	$(\mathbf{1},\mathbf{1})$, $(\mathbf{8}_+,\mathbf{1})$	(0^{16}), $(\underline{\pm\frac{1}{2}^4},-\frac{1}{2}^{12})$				
5	$\mathbf{15}$	$(\frac{1}{2},\underline{-\frac{1}{2}^{14},\frac{1}{2}})$	$\mathbf{1}$	$(-\frac{1}{2}^{16})$		

Table 1: The twisted states and their weights are summarized for the five $\mathrm{SO}(32-6n)\times\mathrm{SU}(3n)$ models, dropping the multiplicity of 27 due to the fixed points.

In table 1 we have collected the representations of the twisted states and indicated their weights. By exploiting spectral flow, we have chosen to use the vector $\tilde{v}$ instead of the original shift vector v to obtain single forms for the weights in this table. (Details concerning spectral flow will be explained in section 4.2.) The U(1) charges of the twisted states are computed as $q\cdot(w+\tilde{v})$. (If one use the original v the corresponding weights will be different but the resulting charge is the same.) The anomaly polynomial due to the twisted states in the representation $(\mathbf{r},\mathbf{s})_q$ reads

$$I_{6|(\mathbf{r},\mathbf{s})_q}=27\Big[-\frac{1}{6}|\mathbf{r}|\mathrm{tr}_{\mathbf{s}}F_{\mathrm{SU}}^3-\frac{1}{6}|\mathbf{r}||\mathbf{s}|q^3F^3+\frac{1}{48}|\mathbf{r}||\mathbf{s}|qF\mathrm{tr}R^2-\frac{1}{2}|\mathbf{r}|q\mathrm{tr}_{\mathbf{s}}F_{\mathrm{SU}}^2-\frac{1}{2}|\mathbf{s}|q\mathrm{tr}_{\mathbf{r}}F_{\mathrm{SO}}^2\Big]. \tag{2.7}$$

For the holonomy triplets there is an extra multiplicity factor of three.

In order that the anomalies can be canceled by the Green-Schwarz mechanism, it is necessary that the total anomaly polynomial I_6 factorizes as

$$I_6=c\,FX_{4|4D}=c\,F\Big[\mathrm{tr}\,R^2-\mathrm{tr}\,F_{\mathrm{SO}}^2-2\mathrm{tr}\,F_{\mathrm{SU}}^2-6n\,F^2\Big]. \tag{2.8}$$

The relative coefficients of the traces are fixed because the four dimensional Green-Schwarz mechanism is remnant of this mechanism in ten dimensions, where the field strength H of the anti-symmetric tensor B fulfills the anomalous Bianchi identity $\mathrm{d}H=X_4=\mathrm{tr}\,R^2-\mathrm{tr}\,F_{\mathrm{SO}(32)}^2$. At the four dimensional fixed points this four form is restricted to $X_{4|4D}$ given in (2.8). The presence of the factor of 2 in front of the $\mathrm{SU}(3n)$ trace is obtained by taking to account the indices of SO and SU groups. This in particular fixes the coefficient in front of the F^2 term: It is given as the trace of the SO(2) generator identified by q, which using the U(1) — rather than the SO(2) — normalization, gives the factor $2\cdot 3n$. The coefficients c are tabulated in table 3. In the $n=5$ model there are two U(1)'s, so that the factorization takes the form

$$I_{6|n=5}=(c\,F+c'\,F')\Big[\mathrm{tr}\,R^2-2\mathrm{tr}\,F_{\mathrm{SU}}^2-6n\,F^2-2\,F'^2\Big]. \tag{2.9}$$

The appropriate coefficients (c,c') are also displayed in table 3. Observe that from this factorization it follows that of the linear combinations of U(1) generators

$$q_a=225q-27q'\,,\qquad q_n=27q'+225q\,, \tag{2.10}$$

only q_a is anomalous.

n	gauge group	untwisted (x 3)	twisted (x 27)
0	SO(32)		$9(\mathbf{1})$
1	$\mathrm{SO}(26)\times\mathrm{SU}(3)\times\mathrm{U}(1)$	$(\mathbf{26},\mathbf{3})_1+(\mathbf{1},\mathbf{3})_{-2}$	$3(\mathbf{1},\overline{\mathbf{3}})_0+(\mathbf{1},\mathbf{1})_{-2}+(\mathbf{26},\mathbf{1})_1$
2	$\mathrm{SO}(20)\times\mathrm{SU}(6)\times\mathrm{U}(1)$	$(\mathbf{20},\mathbf{6})_1+(\mathbf{1},\overline{\mathbf{15}})_{-2}$	$3(\mathbf{1},\mathbf{1})_2+(\mathbf{1},\overline{\mathbf{15}})_0$
3	$\mathrm{SO}(14)\times\mathrm{SU}(9)\times\mathrm{U}(1)$	$(\mathbf{14},\mathbf{9})_1+(\mathbf{1},\overline{\mathbf{36}})_{-2}$	$(\mathbf{1},\overline{\mathbf{9}})_2$
4	$\mathrm{SO}(8)\times\mathrm{SU}(12)\times\mathrm{U}(1)$	$(\mathbf{8},\mathbf{12})_1+(\mathbf{1},\overline{\mathbf{66}})_{-2}$	$(\mathbf{1},\mathbf{1})_4+(\mathbf{8}_+,\mathbf{1})_{-2}$
5	$\mathrm{SU}(15)\times\mathrm{U}(1)\times\mathrm{U}(1)'$	$(\mathbf{15})_{1,-1}+(\mathbf{15})_{1,1}+(\overline{\mathbf{105}})_{-2,0}$	$3(\mathbf{1})_{-\frac{5}{2},-\frac{1}{2}}+(\mathbf{15})_{-\frac{3}{2},\frac{1}{2}}$

Table 2: This table gives the complete spectrum of the SO(32) heterotic $\mathbb{Z}_3$ orbifold models with the gauge shift vector $v=\frac{1}{3}(0^{16-3n},1^{2n},-2^n)$.

n	1	2	3	4	5
c	18	9	$\frac{27}{2}$	-9	$(-\frac{225}{8},\frac{27}{8})$

Table 3: The factorization coefficients defined in (2.8) are tabulated for the five $\mathbb{Z}_3$ orbifold models. As the theory $n=5$ contains two U(1)'s there in total four coefficients; the factorization in that case is given in (2.9).

3. $\mathbb{Z}_4$ models

Next we study the heterotic SO(32) theory on $T^6/\mathbb{Z}_4$. The analysis is to a large extent similar to the $\mathbb{Z}_3$ case except that, as $T^6/\mathbb{Z}_4$ is an even order orbifold, it has six dimensional hyper surfaces at the fixed points of the orbifold $T^4/\mathbb{Z}_2$. Four of them are orbifolds $T^2/\mathbb{Z}_2$ and the other 12 are T^2's that are mapped pairwise to each other. The fixed points of the four $T^2/\mathbb{Z}_2$ combined coincides with the fixed points of the original $T^6/\mathbb{Z}_4$. The situation is very similar to the heterotic E_8 theory on the same orbifold studied in ref. [49]. The spacetime and gauge shift vectors are given by

$$\phi=\frac{1}{4}(1^2,-2)\,,\qquad v=\frac{1}{4}(0^{n_0},1^{n_1},2^{n_2})\,,\tag{3.1}$$

with $n_0=16-n_1-n_2$. (We ignore the possibility of having spinorial shifts of the form $\frac{1}{8}(1^{n_1},3^{n_3},5^{n_5},7^{n_7})$.) For the shift vector v the resulting six and four dimensional gauge groups are

$$\mathrm{SO}(32)\to\mathrm{SO}(2(n_0+n_2))\times\mathrm{SO}(2n_1)\to\mathrm{SO}(2n_0)\times\mathrm{U}(n_1)\times\mathrm{SO}(2n_2)'\,.\tag{3.2}$$

Notice that by taking $n_2\to16-n_1-n_2$ the unbroken gauge group is mapped to itself, therefore we may restrict $2n_2\leq16-n_1$ (on the level of the gauge shift this equivalence is achieved by adding a spinor weight to the shift vector v). The constraint of modular invariance in four dimensions gives

$$2(v^2-\phi^2)=\frac{1}{8}n_1+\frac{1}{2}n_2-\frac{3}{4}\equiv0\,.\tag{3.3}$$

If this condition is satisfied, the modular invariance requirement in six dimension $(2v)^2-(2\phi)^2\equiv0$ is fulfilled as well, and hence does not give additional constraints. There are two independent set of solutions to this: $(n_1, n_2) = (2 + 8p_1, 1 + 2p_2)$ with $p_1 = 0, 1$ and $0 \le p_2 \le 3 - 2p_1$, and $(n_1, n_2) = (6 + 8p_1, 2p_2)$ with $p_1 = 0, 1$ and $0 \le p_2 \le 2 - 2p_1$. This constitutes a total of ten independent models in four dimensions, while on the six dimensional hyper surfaces we encounter only a choice of two massless spectra.

Let us describe the six dimensional spectrum that correspond to the orbifold $T^4/\mathbb{Z}_2$ with gauge shift $2v$. The untwisted matter,

$$\mathbf{R}: \quad 2v \cdot w - \frac{1}{2} \equiv 0: \quad (\underline{\pm 1, 0^{n_1 - 1}}, \underline{\pm 1, 0^{15 - n_1}})\,, \tag{3.4}$$

forms the representation $(\mathbf{2n_1}, \mathbf{2(16 - n_1)})$ of the six dimensional gauge group. In addition there are two types of twisted matter

$$\mathbf{D}: \quad \frac{1}{2}(w + v_2)^2 = \frac{1}{4}\,, \qquad \text{or} \qquad \mathbf{S}: \quad \frac{1}{2}(w + v_2)^2 = \frac{3}{4}\,, \tag{3.5}$$

where $v_2 = \frac{1}{2}(0^{n_0}, 1^{n_1}, 0^{n_2})$. The hyper multiplets $\mathbf{D}$ have a multiplicity of 20: As these states are obtained from the vacuum by acting with the oscillator $\alpha^i_{-1/2}, \alpha^{\underline{i}}_{-1/2}$ with $i, \underline{i} = 1, 2$ which gives a factor of 2 when hyper multiplets are counted. As observed above, 12 of the 16 fixed points of $T^4/\mathbb{Z}_2$ are mapped to each other, while the 4 four are inert under the residual $\mathbb{Z}_4$ action, this gives an additional multiplicity factor of $6 + 4 = 10$. The states in $\mathbf{S}$ are pseudo real and form so-called half-hyper multiplets and they get the same fixed point multiplicity factor 10, but count only half. In table 4 we give the full six dimensional spectrum.

We turn to the four dimensional spectrum. The untwisted matter now comes in two varieties [50]:

$$\mathbf{R}_{i=1,2}: \quad v \cdot w - \frac{1}{4} \equiv 0: \begin{cases} (\mathbf{2n_0}, \mathbf{n_1}, \mathbf{1})_1 : (\underline{\pm 1, 0^{n_0 - 1}}, \underline{1, 0^{n_1 - 1}}, 0^{n_2})\,, \\ (\mathbf{1}, \overline{\mathbf{n_1}}, \mathbf{2n_2})_0 : (0^{n_0}, \underline{-1, 0^{n_1 - 1}}, \underline{\pm 1, 0^{n_2 - 1}})\,, \end{cases} \tag{3.6}$$

$$\mathbf{R}_{i=3}: \quad v \cdot w - \frac{1}{2} \equiv 0: \begin{cases} (\mathbf{1}, \tfrac{\mathbf{1}}{\mathbf{2}}\mathbf{n_1}(\mathbf{n_1} - \mathbf{1}), \mathbf{1})_2 : (0^{n_0}, \underline{1^2, 0^{n_1 - 2}}, 0^{n_2})\,, \\ (\mathbf{1}, \overline{\tfrac{\mathbf{1}}{\mathbf{2}}\mathbf{n_1}(\mathbf{n_1} - \mathbf{1})}, \mathbf{1})_{-2} : (0^{n_0}, \underline{-1^2, 0^{n_1 - 2}}, 0^{n_2})\,, \\ (\mathbf{2n_0}, \mathbf{1}, \mathbf{2n_2})_0 : (\underline{\pm 1, 0^{n_0 - 1}}, 0^{n_1}, \underline{\pm 1, 0^{n_2 - 1}})\,, \end{cases} \tag{3.7}$$

The former has a multiplicity of two, while the latter does not contribute to anomalies as it consists of vector-like representations only. In table 4 we have collected the untwisted matter $\mathbf{R}_{i=1,2}$ that can contribute to anomalies only.

The next part of the four dimensional spectrum consists of the zero modes of the six dimension states. Since only four fixed points of $T^4/\mathbb{Z}_2$ are left invariant by the $\mathbb{Z}_4$ action, only the states on the corresponding T^2 are orbifolded and can give rise to a chiral four dimensional spectrum. Instead, the six dimensional states on the fixed points that are mapped to each other, giving a vector-like zero mode spectrum in four dimensions. In table 4 we only give the chiral spectra that arises from a single fixed point of $T^2/\mathbb{Z}_2$. The four dimensional spectrum is completed by four dimensional twisted states, that are determined by

$$\mathbf{T}: \quad \frac{1}{4}(w + v)^2 = 0, \frac{1}{4}, \frac{1}{2}\,, \tag{3.8}$$

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n_1	6D gauge group	6D untwisted $\mathbf{R}$	6D twisted $\mathbf{D},\mathbf{S}$	
n_2	4D gauge group	4D untwisted $\mathbf{R}_{i=1,2}$	6D twisted on $T^2/\mathbb{Z}_2$	4D twisted $\mathbf{T}$
2	SO(28) × SO(4)	$(\mathbf{28},\mathbf{4})+4(\mathbf{1},\mathbf{1})$	$20(\mathbf{1},\mathbf{2}_-)+5(\mathbf{28},\mathbf{2}_+)$	
1	SO(26) × U(2) × U(1)′	$(\mathbf{26},\mathbf{2})_{1,0}+(\mathbf{1},\overline{\mathbf{2}})_{-1,\pm1}$	$2(\mathbf{1},\mathbf{2})_{0,0}+(\mathbf{26},\mathbf{1})_{0,0}+(\mathbf{1},\mathbf{1})_{-1,\pm1}$	$5(\mathbf{1},\mathbf{1})_{\frac{1}{2},\frac{1}{2}}+(\mathbf{1},\overline{\mathbf{2}})_{-\frac{1}{2},-\frac{1}{2}}+(\mathbf{1},\mathbf{1})_{-\frac{3}{2},\frac{1}{2}}+(\mathbf{26},\mathbf{1})_{\frac{1}{2},-\frac{1}{2}}$
3	SO(22) × U(2) × SO(6)′	$(\mathbf{22},\mathbf{2},\mathbf{1})_1+(\mathbf{1},\overline{\mathbf{2}},\mathbf{6})_{-1}$	$2(\mathbf{1},\mathbf{2},\mathbf{1})_0+(\mathbf{22},\mathbf{1},\mathbf{1})_{-1}+(\mathbf{1},\mathbf{1},\mathbf{6})_0$	$2(\mathbf{1},\mathbf{1},\mathbf{4}_+)_{\frac{1}{2}}+(\mathbf{1},\mathbf{2},\mathbf{4}_-)_{-\frac{1}{2}}$
5	SO(18) × U(2) × SO(10)′	$(\mathbf{18},\mathbf{2},\mathbf{1})_1+(\mathbf{1},\overline{\mathbf{2}},\mathbf{10})_{-1}$	$2(\mathbf{1},\mathbf{2},\mathbf{1})_0+(\mathbf{18},\mathbf{1},\mathbf{1})_0+(\mathbf{1},\mathbf{1},\mathbf{10})_{-1}$	$(\mathbf{1},\mathbf{1},\mathbf{16}_+)_{\frac{1}{2}}$
7	SO(14) × U(2) × SO(14)′	$(\mathbf{14},\mathbf{2},\mathbf{1})_1+(\mathbf{1},\overline{\mathbf{2}},\mathbf{14})_{-1}$	$2(\mathbf{1},\mathbf{2},\mathbf{1})_0+(\mathbf{14},\mathbf{1},\mathbf{1})_{-1}+(\mathbf{1},\mathbf{1},\mathbf{14})_0$	
6	SO(20) × SO(12)	$(\mathbf{20},\mathbf{12})+4(\mathbf{1},\mathbf{1})$	$5(\mathbf{1},\mathbf{32}_+)$	
0	SO(20) × U(6)	$(\mathbf{20},\mathbf{6})_1$	$(\mathbf{1},\mathbf{1})_{-3}+(\mathbf{1},\overline{\mathbf{15}})_1$	$5(\mathbf{1},\mathbf{1})_{\frac{3}{2}}+(\mathbf{1},\overline{\mathbf{15}})_{-\frac{1}{2}}$
2	SO(16) × U(6) × SO(4)′	$(\mathbf{16},\mathbf{6},\mathbf{1})_1+(\mathbf{1},\overline{\mathbf{6}},\mathbf{4})_{-1}$	$(\mathbf{1},\mathbf{1},\mathbf{1})_3+(\mathbf{1},\mathbf{15},\mathbf{1})_{-1}$	$2(\mathbf{1},\mathbf{1},\mathbf{2}_+)_{\frac{3}{2}}+(\mathbf{1},\overline{\mathbf{6}},\mathbf{2}_-)_{\frac{1}{2}}$
4	SO(12) × U(6) × SO(8)′	$(\mathbf{12},\mathbf{6},\mathbf{1})_1+(\mathbf{1},\overline{\mathbf{6}},\mathbf{8}_v)_{-1}$	$(\mathbf{1},\mathbf{1},\mathbf{1})_{-3}+(\mathbf{1},\overline{\mathbf{15}},\mathbf{1})_1$	$(\mathbf{1},\mathbf{1},\mathbf{8}_+)_{\frac{3}{2}}$
10	SO(12) × SO(20)	$(\mathbf{12},\mathbf{20})+4(\mathbf{1},\mathbf{1})$	$5(\mathbf{32}_-,\mathbf{1})$	
1	SO(10) × U(10) × U(1)′	$(\mathbf{10},\mathbf{10})_{1,0}+(\mathbf{1},\overline{\mathbf{10}})_{-1,\pm1}$	$(\mathbf{16}_-,\mathbf{1})_{0,\frac{1}{2}}$	$2(\mathbf{1},\mathbf{1})_{\frac{5}{2},\frac{1}{2}}+(\mathbf{1},\overline{\mathbf{10}})_{\frac{3}{2},-\frac{1}{2}}$
3	SO(6) × U(10) × SO(6)′	$(\mathbf{6},\mathbf{10},\mathbf{1})_1+(\mathbf{1},\overline{\mathbf{10}},\mathbf{6})_{-1}$	$(\mathbf{4}_-,\mathbf{1},\mathbf{4}_+)_0$	$(\mathbf{1},\mathbf{1},\mathbf{4}_+)_{\frac{5}{2}}+(\mathbf{4}_-,\mathbf{1},\mathbf{1})_{-\frac{5}{2}}$
14	SO(4) × SO(28)	$(\mathbf{4},\mathbf{28})$	$20(\mathbf{2}_+,\mathbf{1})+5(\mathbf{2}_-,\mathbf{28})$	
0	SO(4) × U(14)	$(\mathbf{4},\mathbf{14})_1$	$2(\mathbf{2}_+,\mathbf{1})_0+(\mathbf{2}_-,\overline{\mathbf{14}})_{-1}$	$(\mathbf{1},\mathbf{1})_{\frac{7}{2}}+(\mathbf{2}_+,\mathbf{1})_{-\frac{7}{2}}$

Table 4: The six and four dimensional gauge groups are tabulated of the SO(32) heterotic $\mathbb{Z}_4$ orbifold models defined by the gauge shift vector $v=\frac{1}{4}\big(0^{n_0},1^{n_1},2^{n_2}\big)$, $n_0=16-n_1-n_2$. The six dimensional (half) hyper multiplets included the multiplicity factors that count the number of independent $T^4/\mathbb{Z}_2$ fixed points within $T^6/\mathbb{Z}_4$. The four dimensional twisted states and zero modes of the twisted states on $T^2/\mathbb{Z}_2$ complete the table. This table does not give the complete four dimensional spectrum, only the chiral part, relevant for anomaly considerations.

and are also given in table 4. Having determined all possible $\mathbb{Z}_4$ models and spectra in four dimensions, we can read off the chiral spectrum from table 4. (In this table we have not given the vector-like representations since they are not relevant for anomaly considerations nor for phenomenology, since they can easily acquire large mass.)

There are two models that may be interesting in the context of GUT model building: The models with $(n_1, n_2) = (2, 5)$ and $(10, 1)$ contain SO(10) factors, and both models contain **16** spinor representations of SO(10), which can accommodate full generations of quarks and leptons including right-handed neutrinos. The models do not have an equal number of generations because the origin of these spinor representations is different: The $(2, 5)$ model has 16 generations because the spinors arise as four dimensional twisted states at the 16 fixed points of $T^6/\mathbb{Z}_4$. The other model is more interesting from the point of view of phenomenology since it only has four generations. The spinor **16** is obtained from the orbifolding of the six dimensional twisted states that reside at the four fixed points of $T^4/\mathbb{Z}_2$ that are left inert by the residual $\mathbb{Z}_4$ action. Both models suffer from the usual difficulty that the Higgs sector is not rich enough to give rise to symmetry breaking down to the SM. Further symmetry breaking can of course be enforced by the inclusion of Wilson lines and then this model may be a promising starting point for an orbifold GUT.

4. Classification of orbifold models

As the number of $\mathbb{Z}_3$ and $\mathbb{Z}_4$ models was still relatively small the classification could be performed by hand. For arbitrary six dimensional $\mathbb{Z}_7$ or $\mathbb{Z}_{2N}$ $(N = 2, 3, 4, 6)$ orbifolds this becomes a formidable task to be performed by a computer, unless some classification systematics is developed both to identify the modular invariant shifts and to determine the twisted states. Here we describe efficient methods to do both and illustrate them with the $\mathbb{Z}_3$ and $\mathbb{Z}_4$ models discussed in section 2 and 3. There are different classes of models depending on the choice of the spatial shift ϕ. Moreover, the geometry of fixed hyper surfaces plays an important role in how the final four dimensional matter spectrum is composed. Therefore we have organized this section as follows: First we explain the geometrical structure of the hyper surfaces within a given $\mathbb{Z}_{2N}$ orbifold. Next we give a complete classification of all possible gauge shift vectors. After that we compute the twisted matter located at the fixed hyper surfaces of the orbifold. Finally we combine the matter spectra at the various fixed points to identify the full six and four dimensional zero mode spectrum of the theory.

The geometrical properties of $T^6/\mathbb{Z}_{2N}$ orbifolds are more complicated than the prime orbifold $T^6/\mathbb{Z}_3$, see [7, 13] for a more detailed discussion. In a $T^6/\mathbb{Z}_{2N}$ orbifold we can distinguish the hyper surfaces that are left fixed by the orbifold action of $\mathbb{Z}_M \subset \mathbb{Z}_{2N}$ subgroups by their dimensions. The number of entries of the spatial shift $p\phi$ of the $\mathbb{Z}_M$ subgroup that are non-vanishing modulo one gives the complex dimensionality. The form of the spatial shift vector ϕ of $\mathbb{Z}_{2N}$ required by supersymmetry implies that these dimensions can be either four or six. In either case there is a residual $\mathbb{Z}_{2N}/\mathbb{Z}_M$ action of the full orbifold group $\mathbb{Z}_{2N}$ on any hyper surface fixed by $\mathbb{Z}_M$. This leads to an identification of all $\mathbb{Z}_M$ hyper surfaces that are mapped to each other by this residual group action, or to further orbifolding. If the dimension of the hyper surface is six, a two torus T^2 is left inert under the $\mathbb{Z}_M$ subgroup, and the residual action gives rise to the orbifold $T^2/(\mathbb{Z}_{2N}/\mathbb{Z}_M)$. The orbifolding of this T^2 can be understood using field theoretical methods as we explain in subsection 4.4.

We restrict our explicit classification resulting in the tables of the appendices to models with only one subgroup that is $\mathbb{Z}_2$, giving six-dimensional hyper surfaces, to keep our paper at moderate length. This means that the tables in the appendices describe the $\mathbb{Z}_6$-I and $\mathbb{Z}_8$-I models, with spatial shifts given in (1.3), only. The six dimensional orbifold geometry is always of the form $T^4/\mathbb{Z}_2$ with 16 fixed points. The $\mathbb{Z}_4$ orbifold, which we already described in the previous section, indeed follows these general patterns: The only non-trivial subgroup of $\mathbb{Z}_4$ in that case is $\mathbb{Z}_2$. The twelve of the $\mathbb{Z}_2$ fixed tori T^2 are mapped each other by the residual $\mathbb{Z}_2$ action, leaving 6 independent tori. While the other four fixed points of $T^4/\mathbb{Z}_2$ are also fixed points of the $\mathbb{Z}_4$ action giving rise to two-dimensional orbifolds $T^2/\mathbb{Z}_2$.

Odd order orbifolds can be treated in exactly the same way as the even order orbifolds when computing the twisted spectra. We show this by revisiting the $\mathbb{Z}_3$ twisted states in subsection 4.3.1. Only the classification of modular invariant shifts requires slightly more care. In section 2 we have exhausted all possibilities for $\mathbb{Z}_3$ orbifolds, the $\mathbb{Z}_7$ case is discussed in appendix B.

4.1 Classification of modular invariant $\mathbb{Z}_{2N}$ shifts

The classification of $\mathbb{Z}_{2N}$ modular invariant models can be done on the level of their defining gauge shifts only. For a $\mathbb{Z}_{2N}$ gauge shift we may consider two types of gauge shift vectors that we refer to as vectorial and spinorial shifts. For the sake of simplicity we restrict to the vectorial ones only. The spinorial shifts case can be explored straightforwardly, see subsection 5.1. Any vectorial shift can be brought to the form

$$v = \frac{1}{2N}\left(0^{n_0}, \ldots, N^{n_N}\right), \qquad \text{with} \sum_{k=0}^{N} n_k = 16\,, \tag{4.1}$$

and lead to the symmetry breaking pattern

$$\mathrm{SO}(32) \to \mathrm{SO}(2n_0) \times \mathrm{U}(n_1) \times \cdots \times \mathrm{U}(n_{N-1}) \times \mathrm{SO}(2n_N) \tag{4.2}$$

in four dimensions. In case of confusion, like with the $\mathrm{U}(1)$ factors, we employ a subscript to make the distinction between the various factors, for example $\mathrm{U}(n_k) = \mathrm{U}(1)_k \times \mathrm{SU}(n_k)$ and $\mathrm{U}(1)_0 = \mathrm{SO}(2n_0)$ when $n_0 = 1$.

The form of the vectorial shift vectors introduced above constitutes the most general choice from a field theoretical point of view. In string theory only those shift vectors are allowed that lead to a modular invariant theory, i.e. a theory that satisfies the $\mathbb{Z}_{2N}$ level matching condition

$$N(\phi^2 - v^2) \equiv 0\,, \tag{4.3}$$

see (A.5) of appendix A. Written in terms of (4.1) the level matching condition is

$$N\,\phi^2 \equiv N\,v^2 = \frac{1}{4N}\sum_{k=1}^{N} k^2\, n_k\,. \tag{4.4}$$

In the shift vector (4.1) we restrict the values of its entries shift vector to $0,\ldots,N/(2N)$. This is allowed since the entries $p/(2N)$ and $(p-2N)/(2N)$ results in the same contribution to level matching condition:

$$N\left(\frac{p}{2N}\right)^2 \equiv N\left(\frac{p-2N}{2N}\right)^2. \tag{4.5}$$

The signs in the shift vector (4.1) are also not relevant as one can switch the signs of the weight entries correspondingly. We can view $v = v_n$ as a function of vector $n = (n_1,\ldots,n_N)$, so that $N\,v^2$ defines a linear function of n and $N\,v_{n+n'}^2 = N\,v_n^2 + N\,v_{n'}^2$, where the shift $v_{n+n'}$ is defined in the obvious way. We call v_ν a null-shift if $N\,v_\nu^2 \equiv 0$. Notice that if v_n and $v_{n'}$ are solutions of the modular invariance requirement (4.4), then $v_{n-n'}$ is a null-shift. Hence any solution of the level matching condition can be obtained as $v_{n+\nu}$ with v_n any fixed solution of (4.4) and an null-shift v_ν.

This method allows for a simple classification since a particular base solution of the level matching condition (4.4) is easy to guess, and the null-solutions are obtained using elementary linear algebra. To exemplify this, we return the $\mathbb{Z}_4$ case studied in section 3. The null-solutions v_ν for the level matching condition (3.3) are identified by

$$\nu = (\nu_1,\nu_2) = (8p_1, 2p_2) + (4,-1)q\,, \tag{4.6}$$

with p_1, p_2, q integers. As base solution we can choose v_n with $n = (6,0)$. It is not difficult to see that any solution described in section 3 is given by $v_{n+\nu}$. In the appendices we preform this classification of the modular invariant shift for the $\mathbb{Z}_6$-I, $\mathbb{Z}_8$-I and $\mathbb{Z}_7$ models.

4.2 Irreducible twisted spectra on $\mathbb{C}^d/\mathbb{Z}_M$

The next task is to compute the full local spectra of matter states at the fixed points. The untwisted sector can be obtained by orbifolding the original SO(32) gauge theory coupled to $\mathcal{N}=1$ supergravity in ten dimensions. As this can be understood by group theoretical methods in field theory, it will be postponed to the final subsection 4.4 of the present section. Not only the untwisted sector can be understood using field theoretical orbifolding, also the orbifolding of the $\mathbb{Z}_M$ twisted states by the residual group $\mathbb{Z}_{2N}/\mathbb{Z}_M$ can be analyzed this way. For this reason we focus our attention to irreducible twisted string spectra, i.e. spectra that are not obtained by orbifolding untwisted or twisted states. Moreover, to compute the irreducible twisted spectra at a given $\mathbb{Z}_M$ fixed point, it is irrelevant that the full orbifold is compact or not, this allows us to study the twisted states on $\mathbb{C}^d/\mathbb{Z}_M$, which have just a single fixed point at 0. By taking the complex dimension $d = 2, 3$ we can model the fixed points of $T^4/\mathbb{Z}_M$ and $T^6/\mathbb{Z}_M$, respectively. The requirement that we only consider irreducible string spectra can be translated to the condition that the pth twisted

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sector of $\mathbb{Z}_M$ is only taken into account if p is relatively prime with M. Next we move to the technical details of the classification of twisted states.

For the classification of the irreducible twisted states it is important to know the chirality of these states. The six dimensional twisted states form hyper multiplets and therefore have opposite chirality to that of gauginos, because of the strong constraints of six dimensional supersymmetry. In four dimensions the chirality of the fermions in twisted chiral multiplets can be either left- or right-handed. For physical investigations, like anomaly considerations, it is convenient to fix the chirality of all chiral multiplets in the same way. The chirality $\Sigma_p = \pm 1$ of the pth twisted sector is determined by

$$\frac{1}{4}\Sigma_p \equiv -\frac{1}{2}\sum_i \tilde{\phi}_{p\,i}\,, \tag{4.7}$$

where $\tilde{\phi}_{p\,i}$ is defined in (4.10), see [51]. The chirality of the $(M-p)$th and the pth twisted sectors are opposite: $\Sigma_{M-p} = -\Sigma_p$. Therefore, if we combine the twisted states in chiral multiplets we only need to take the pth or the $(M-p)$th twisted sector into account. Since the chirality of the untwisted and the first twisted sector is always positive, we always select the positive chirality to determine the chiral multiplet representation.

We investigate what kind of representations will be encountered in heterotic SO(32) orbifold theories. The irreducible twisted matter representations is described in terms of tensor products of irreducible representations of U(n) and SO($2n$). Because of the two spin-structures of the gauge fermions on the worldsheet theory we encounter both vectorial and spinorial weights as is well-known. A derivation of the appropriate mass, GSO and orbifold conditions is reviewed in appendix A. The objective of this section is to give a complete classification of all possible representations that ever arise, to this end it is extremely useful to fix form of both the vectorial and spinorial weights to:

$$\tilde{w} = (1, \ldots, -1, \ldots, 0, \ldots)\,, \tag{4.8}$$

with all possible permutations. This standard form of the weights makes general patterns of irreducible representations transparent as it automatically takes spectral flow into account: Normally, one specifies a shift vector and then determines the weights corresponding to the massless states, but then the weights of states within the same representation can be very different one from another. Physically this does not constitute a problem since the weight is just a label for a state. However, for the general classification we aim at in this paper, it is a nuisance, which is easily avoided by redefining the shift vector $v \to \tilde{v}$ by adding an integral vector to v. We show explicitly that one is allowed to do so in appendix A using the string partition function. To ensure that only these standard weights (4.8) can satisfy the massless condition for the pth twisted sector

$$\frac{1}{2}(\tilde{w} + \tilde{v}_p)^2 = \frac{5}{8} + \frac{1}{2}\tilde{\phi}_p^2 - \tilde{N}\,, \qquad \tilde{N} = \sum_i \left(\frac{1}{2} - s_i\tilde{\phi}_{p\,i}\right) r_i\,, \tag{4.9}$$

(where the sum is over all entries of $\phi_i \not\equiv 0$) we need that the entries of $\tilde{v}_p = \tilde{v}_p^{\text{vec}}, \tilde{v}_p^{\text{spin}}$ and $\tilde{\phi}_p$ lie between: $-\frac{1}{2} < \tilde{v}_{p\,I}, \tilde{\phi}_{p\,i} \leq \frac{1}{2}$. This requirement uniquely determines the integral

$\tilde{v}$	group	repr.	weights	prop.	mass: $\frac{1}{2}(w+\tilde{v}\,e_n)^2$
$\tilde{v}=0$	SO($2n$)	$[\mathbf{2n}]^k$	$(\underline{\pm 1^k, 0^{n-k}})$	$k=0,1$	$\frac{k}{2}$
$0<\lvert\tilde{v}\rvert<\frac{1}{2}$	U(n)	$[\mathbf{n}]_k^\alpha$	$\alpha(\underline{1^k, 0^{n-k}})$	$\alpha=\pm$ $k\geq 0$	$k(\frac{1}{2}+\alpha\tilde{v})+\frac{n}{2}\tilde{v}^2$
$\tilde{v}=\frac{1}{2}$	SO($2n$)	$\mathbf{2}_\alpha^{\mathbf{n-1}}$	$(\underline{-\frac{1}{2}^k, \frac{1}{2}^{n-k}})-\frac{1}{2}e_n$	$\alpha=(-)^k$	$\frac{n}{8}$

Table 5: The twisted matter of SO(32) orbifold models are built out of the following representations: the k-form representations $[\mathbf{n}]_k^\pm$ of U(n) and the vector $\mathbf{2n}$ and spinor $\mathbf{2}_\pm^{\mathbf{n-1}}$ representations of SO($2n$).

vectors $d_p^{\text{vec}}, d_p^{\text{spin}} \in \mathbb{Z}^{16}$ in the definitions of shifts

$$\tilde{v}_p^{\text{vec}} = (pv - d_p^{\text{vec}})S_p\,, \qquad \tilde{v}_p^{\text{spin}} = (pv - \frac{1}{2}e - d_p^{\text{spin}})S_p\,, \qquad \tilde{\phi}_p = p\phi + \tfrac{1}{2}e_3 + \delta_p\,, \quad (4.10)$$

where $e = (1^{16})$ and $e_3 = (1,1,1)$. To ensure that all entries of $\tilde{v}_p^{\text{vec}}$ are positive and ordered:

$$\tilde{v}_p^{\text{vec}} = \frac{1}{M}\big(0^{n_0}, \ldots, m^{n_m}\big)\,, \quad (4.11)$$

where $m = [M/2]$ is the integral part of $M/2$, we have introduced also the matrix S_p that only has entries equal to ± 1 and 0. (Up to this matrix the vectors $\tilde{v}_p^{\text{vec}}$ and $\tilde{v}_p^{\text{spin}}$ are the same as the shifts v_p^{vec} and v_p^{spin} introduced in appendix A.) The mass contribution due to the bosonic oscillators is denoted by $\tilde{N}$: The signs $s_i = \pm$ and integers $r_i \geq 0$ indicate that r_i bosonic oscillators with internal space time index i (if $s_i = +$) or $\underline{i}$ (if $s_i = -$) have been applied to the vacuum state. Not all states that satisfy the mass condition (4.9) are present in the physical spectrum, only those states survive that fulfill the GSO and orbifold projections:

$$\begin{aligned} \text{GSO}: \quad & \frac{1}{2}e\cdot\tilde{w} \equiv \frac{1}{2}e\cdot d_p\,, \\ \text{Orbifold}: \quad & vS_p\cdot\tilde{w} \equiv \frac{1}{2}(v^2-\phi^2)p - vS_p\cdot\tilde{v}_p + \sum_i \phi_i(s_i r_i + \tilde{\phi}_{p\,i})\,, \end{aligned} \quad (4.12)$$

where in the GSO condition we have used that $\frac{1}{2}e\cdot\tilde{w} \equiv \frac{1}{2}e\cdot w$. The orbifold condition is, in fact, obsolete for the irreducible twisted sectors, since one can show that by taking the mass condition (4.9) modulo one and combining it with the GSO projection implies the orbifold projection.

The possible types of gauge groups, that arise in heterotic SO(32) models, are either SO($2n$) or U(n). All U(n) representations are totally anti-symmetric k-form tensor representations of the vector $\mathbf{n}$ or its complex conjugate $\overline{\mathbf{n}}$, denoted by $[\mathbf{n}]_k^+$ and $[\mathbf{n}]_k^-$, respectively. (In particular, $[\mathbf{n}]_0^\pm = \mathbf{1}$, $[\mathbf{n}]_1^+ = \mathbf{n}$, $[\mathbf{n}]_1^- = \overline{\mathbf{n}}$, and $[\mathbf{n}]_k = [\overline{\mathbf{n}}]_{n-k}$.) The representations of SO($2n$) that arise are the fundamental representation $[\mathbf{2n}]^k$ or the spinor representation $\mathbf{2}_\alpha^{\mathbf{n}-1}$ of $\alpha = \pm$ chirality. The index $k = 0,1$ is used to simultaneously treat the fundamental and the singlet representation. In table 5 we have collected the various representations that can arise and indicate to which vectorial and spinorial weights they correspond. Moreover, we have given the value of their mass contribution $\frac{1}{2}(w+\tilde{v}e_n)^2$ with $e_n = (1^n)$. Using these representations we can identify the irreducible twisted states for both the vectorial and spinorial weights. The vectorial weights give rise to representations of the form

$$\mathbf{R}_{\text{vec}} = \left([\mathbf{2n_0}]^{k_0}\,,\,[\mathbf{n_1}]^{\alpha_1}_{k_1}\,,\,\dots\,,\,[\mathbf{n_{m-1}}]^{\alpha_{m-1}}_{k_{m-1}}\,,\,\mathbf{2}^{\mathbf{n_m}-\mathbf{1}}_{\alpha_\mathbf{m}}\right), \tag{4.13}$$

where $\alpha_a = \pm$, $k_0 = 0, 1$ and $k_a \geq 0$. The mass contribution of this state reads

$$\frac{k_0}{2} + \sum_{a=1}^{m-1} k_a\left(\frac{1}{2} + \alpha_a \tilde{v}^{\text{vec}}_{p\,a}\right) + \frac{1}{2}(\tilde{v}^{\text{vec}}_p)^2 = \frac{5}{8} + \frac{1}{2}\tilde{\phi}_p^2 - \tilde{N}\,. \tag{4.14}$$

The GSO projection (4.12) on the vectorial weights require that

$$\frac{1-\alpha_m}{4} + \frac{1}{2}\sum_{a=0}^{m-1} k_a \equiv \frac{1}{2} e \cdot d^{\text{vec}}_p\,. \tag{4.15}$$

The spinorial weights give very similar representations, except that the roles of the spinor and vector representations of the SO groups are interchanged:

$$\mathbf{R}_{\text{spin}} = \left(\mathbf{2}^{\mathbf{n_0}-\mathbf{1}}_{\alpha_\mathbf{0}}\,,\,[\mathbf{n_1}]^{\alpha_1}_{k_1}\,,\,\dots,\,[\mathbf{n_{m-1}}]^{\alpha_{m-1}}_{k_{m-1}}\,,\,[\mathbf{2n_m}]^{k_m}\right), \tag{4.16}$$

where $\alpha_a = \pm$, $k_m = 0, 1$ and $k_a \geq 0$. The mass formula in this case becomes

$$\frac{k_m}{2} + \sum_{a=1}^{m-1} k_a\left(\frac{1}{2} + \alpha_k \tilde{v}^{\text{spin}}_{p\,a}\right) + \frac{1}{2}(\tilde{v}^{\text{spin}}_p)^2 = \frac{5}{8} + \frac{1}{2}\tilde{\phi}_p^2 - \tilde{N}\,, \tag{4.17}$$

and the GSO projection (4.12) on the spinorial weights reads

$$\frac{1-\alpha_0}{4} + \frac{1}{2}\sum_{a=1}^{m} k_a \equiv \frac{1}{2} e \cdot d^{\text{spin}}_p\,. \tag{4.18}$$

Let us make some final comments about the irreducible twisted state representations: Since these are general results, one may obtain anti-symmetric representation $[\mathbf{n}]_k$ with $k > n$, which vanishes identically. We simply drop it all together when it can never be part of the spectrum. The $\mathrm{U}(1)_a$ charge $q_a = (0^{n_0+\cdots}, 1^{n_a}, 0^{\cdots+n_m})$ is computed using the formula $q_a S_p \cdot (\tilde{v} + \tilde{w})$. For example, the q_a charge of the vectorial weight state (4.13) in the first twisted sector reads

$$q_a \cdot \mathbf{R}_{\text{vec}} = (\alpha_a k_a + n_a v_a)\mathbf{R}_{\text{vec}}\,. \tag{4.19}$$

This concludes the description of the representations that arise within a given irreducible twisted sector.

The various twisted sectors are not independent entities, but are closely related. These relations are encoded in the matrices S_p introduced in (4.10) above. The defining property of the matrix S_p is that they bring the gauge shift pv (modulo integers) of the pth twisted sector back to the standard positive ordered form given in (4.11). Now assume that the pth twisted sector is irreducible, then the matrix S_p indicates how the spectrum in this sector can be obtained from the first twisted sector without any calculation: An off-diagonal entry of this matrix with $(S_p)_{ab} = +1$ indicates that in all vectorial and spinorial representations, (4.13) and (4.16), of the first twisted sector one replaces: $\mathbf{n}_b \to \mathbf{n}_a$, while for $(S_p)_{ab} = -1$: $\mathbf{n}_b \to \overline{\mathbf{n}}_a$. This specifies the complete spectrum of the pth twisted sector up to the spinor chiralities that are determined by the GSO conditions. By applying the matrix S_p various times one can generate all irreducible twisted sectors from the first twisted sector. The number of irreducible twisted sectors is given by the smallest K such that $(S_p)^K = \mathbb{1}$. This clearly reduces the effort of obtaining the irreducible twisted spectra, and shows that they are intimately related. For example, in appendix B we derive the twisted sectors of $\mathbb{Z}_7$ models. The matrix S_2 given in (B.5) satisfies $(S_2)^3 = \mathbb{1}$, hence all three twisted sectors can be obtained from the first twisted sector using $S_2, (S_2)^2$.

The matrix S_p is also defined when the p twisted sector is reducible, i.e. for sectors that are obtained by orbifolding of sectors of $\mathbb{Z}_{M/p}$ orbifolds. In this case the matrix S_p indicates how the gauge group in the $\mathbb{Z}_{M/p}$ model is broken to the one in the $\mathbb{Z}_M$ model. Moreover, it indicates whether some groups and their representations are interchanged or complex conjugated. Let us also illustrate this situation with a concrete example: In section 3 we saw that the second twisted sector of $\mathbb{Z}_4$ models is six dimensional. The second branching in (3.2) is encoded by the matrix

$$S_2 = \left(\begin{array}{c|c|c} \mathbb{1}_{n_0} & & \\ \hline & & \mathbb{1}_{n_1} \\ \hline & \mathbb{1}_{n_2} & \end{array}\right). \tag{4.20}$$

In table 4 we see the interchange of the representation, when one goes from the six to the four dimensional representations. This exemplifies the importance of the matrices S_p, and therefore, in the appendices we will present give them explicitly for the various orbifold models.

After having derived the general results that determine the irreducible twisted spectra, we describe a convenient way to represent them in the various models. As we will make extensive use of this representation in the appendices, we illustrate it in subsection 4.3.2. For the twisted matter of $\mathbb{Z}_{2N}$ orbifold models it proves convenient to employ a combination of three tables to display the spectrum. The reason for this is that the spectrum is built out of vectorial and spinorial weights, which receive identical contributions from the bosonic oscillators through $\tilde{N}$ in the mass formulas (4.14) and (4.17). The first table gives the possible values of $\tilde{N}$, the corresponding index structure of these states, their (SU(2)) holonomy representations and total multiplicities. The other two tables give the possible representations for vectorial and spinorial weights as functions of the quantities $\frac{1}{2}\tilde{v}_{\text{vec}}^2$ and $\frac{1}{2}\tilde{v}_{\text{spin}}^2$, respectively. The entries of these two tables are the values of $\tilde{N}$ such that the mass condition is fulfilled. The GSO projection leads to the additional complication, that the chiralities can be model dependent. Moreover, for models with n_0 or n_m is zero the GSO conditions (4.15) and (4.18) projects out those states that would a have the wrong chirality if the corresponding spinor would not be zero. All these aspects will be illustrated by the tables 7 and 8 of subsection 4.3.2 where we derive the first twisted spectrum of $\mathbb{Z}_4$ models.

n	vectorial solutions	repr.
0	$r_i = 1, s_i = +, k_a = 0$	$(\mathbf{1})_{\mathbf{3}}$
	$r_i = 2, s_i = -, k_a = 0$	$(\mathbf{1})_{\overline{\mathbf{6}}}$
1	$r_i = 0, k_0 = 1$	$(\mathbf{26}, \mathbf{1})_{\mathbf{1}}$
	$r_i = 0, k_1 = 3, \alpha_1 = -$	$(\mathbf{1}, [\overline{\mathbf{3}}]_3)_{\mathbf{1}}$
	$r_i = 1, s_i = -, k_1 = 1, \alpha_1 = -$	$(\mathbf{1}, \overline{\mathbf{3}})_{\overline{\mathbf{3}}}$
2	$r_i = 0, k_1 = 2, s_1 = -$	$(\mathbf{1}, [\overline{\mathbf{6}}]_2)_{\mathbf{1}}$
	$r_i = 1, s_i = -, k_a = 0$	$(\mathbf{1}, \mathbf{1})_{\overline{\mathbf{3}}}$
3	$r_i = 0, k_1 = 1, \alpha_1 = -$	$(\mathbf{1}, \overline{\mathbf{9}})_{\mathbf{1}}$
4	$r_i = k_a = 0$	$(\mathbf{1}, \mathbf{1})_{\mathbf{1}}$

n	spinorial solutions	repr.
4	$r_i = 0, k_1 = 0, \alpha_0 = +$	$(\mathbf{2}^{\mathbf{3}}_{+}, \mathbf{1})_{\mathbf{1}}$
5	$r_i = 1, s_i = -, k_1 = 0, \alpha_0 = +$	$(\mathbf{2}^{\mathbf{0}}_{+}, \mathbf{1})_{\overline{\mathbf{3}}}$
	$r_i = 0, k_1 = 1, \alpha_1 = +, \alpha_0 = -$	$(\mathbf{2}^{\mathbf{0}}_{-}, \mathbf{15})_{\mathbf{1}}$

Table 6: The classification procedure for irreducible twisted states described in this section is applied to the first twisted sector of $\mathbb{Z}_3$ models. The subscripts indicate the SU(3) holonomy representations of these states. The resulting spectrum for the vectorial and spinorial weights coincides with that obtained in table 1 by more conventional methods.

4.3 Examples of irreducible twisted spectra

Because of the generality of the description of the irreducible it might seem difficult to apply our formalism in concrete situations. Therefore we present three examples to illustrate the efficiency of our method. In the first two examples we determine the first twisted sectors in the $\mathbb{Z}_3$ and $\mathbb{Z}_4$ models that we have studied in sections 2 and 3, respectively. The $\mathbb{Z}_3$ example illustrates how to apply the spectrum formulas (4.14) and (4.17). Instead, we use the $\mathbb{Z}_4$ case to explain the structure of the tables for twisted states we will use in our classification of the other $\mathbb{Z}_{2N}$ and $\mathbb{Z}_7$ models as well. Our final example determines the universal six dimensional structure of the Nth twisted sector in $\mathbb{Z}_{2N}$ orbifolds.

4.3.1 First twisted sector in $\mathbb{Z}_3$ models

The spectrum of the first twisted sector for the $\mathbb{Z}_3$ orbifold including the standard representation of the weights were given in table 1. Even though we have designed the method exposed in the section to be applied to even order orbifolds, it may also be used for odd order orbifolds, with the only restriction that $n_m = 0$, i.e. models with at most a single SO group. For the vectorial and spinorial weight states we find that the mass conditions (4.14) and (4.17) reduces to

$$\begin{aligned} \text{vec}: \quad & \frac{k_0}{2} + k_1\left(\frac{1}{2} + \frac{\alpha_1}{3}\right) = \frac{1}{6}\left[4 - n - \sum_i (3 + s_i) r_i\right], \\ \text{spin}: \quad & k_1\left(\frac{1}{2} - \frac{\alpha_1}{6}\right) = \frac{1}{24}\left[8n - 32 - 4\sum_i (3 + s_i) r_i\right]. \end{aligned} \tag{4.21}$$

We see that the equation for the vectorial weights only has solutions for $n \leq 4$, while the one for the spinorial weights only for $n \geq 4$. The solutions have been summarized in table 6, and it is then straightforward to confirm they correspond to the representations given in table 1 of section 2. All vectorial states that pass the mass shell condition automatically fulfill the relevant GSO conditions, (4.15) and (4.18), as well. For the spinorial weights the GSO selects the chirality of the spinor representations.

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$\tilde{N}$	0	$\frac{1}{4}$	$\frac{1}{2}$
states	$\lvert 0\rangle$	$\lvert\underline{1}^k\underline{2}^{1-k}\rangle$	$\lvert\underline{1}^k\underline{2}^{2-k}\rangle, \lvert 3\rangle, \lvert\underline{3}\rangle$
SU(2) hol.	$\mathbf{1}$	$\overline{\mathbf{2}}$	$\overline{\mathbf{3}}, \mathbf{1}, \mathbf{1}$
multi.	1	2	5

Table 7: The possible values of $\tilde{N}$ for the first twisted sector of a $\mathbb{Z}_4$ are $0, \frac{1}{4}$ and $\frac{1}{2}$. The index structure of the corresponding states, their SU(2) holonomy representations and total multiplicities are indicated.

vectorial repr. $\setminus$ $n_1 + 4n_2$	6	14	22
$\big(\mathbf{2}_{+}^{\mathbf{n_2}-\mathbf{1}}\big)$	$\frac{1}{2}$	$\frac{1}{4}$	0
$\big(\overline{\mathbf{n_1}}, \mathbf{2}_{-}^{\mathbf{n_2}-\mathbf{1}}\big)$	$\frac{1}{4}$	0	
$\big(\mathbf{2n_0}, \mathbf{2}_{-}^{\mathbf{n_2}-\mathbf{1}}\big), \big([\overline{\mathbf{n_1}}]_2, \mathbf{2}_{+}^{\mathbf{n_2}-\mathbf{1}}\big)$	0		$\tilde{N}$

spinorial repr. $\setminus$ $n_1 + 4n_0$	6	14	22
$\big(\mathbf{2}_{\alpha}^{\mathbf{n_0}-\mathbf{1}}\big)$	$\frac{1}{2}$	$\frac{1}{4}$	0
$\big(\mathbf{2}_{-\alpha}^{\mathbf{n_0}-\mathbf{1}}, \mathbf{n_1}\big)$	$\frac{1}{4}$	0	
$\big(\mathbf{2}_{-\alpha}^{\mathbf{n_0}-\mathbf{1}}, \mathbf{2n_2}\big), \big(\mathbf{2}_{\alpha}^{\mathbf{n_0}-\mathbf{1}}, [\mathbf{n_1}]_2\big)$	0		$\tilde{N}$

Table 8: The tables on the left and right display the possible representations in the first twisted sector of $\mathbb{Z}_4$ models for vectorial and spinorial weights, respectively. The entries of these tables give the values of $\tilde{N}$ which determines the internal space properties of these states as can be read off from table 7. The GSO projection selects the chirality of the spinor representations, which for the spinorial weights depends on $\alpha = (-)^{n_0}$. When n_0 or n_2 is zero the GSO only leaves those states that would have the positive chirality.

4.3.2 First twisted sector in $\mathbb{Z}_4$ models

To illustrate the derivation of the tables we employ in the appendices, we compute the first twisted sector in $\mathbb{Z}_4$ orbifolds. The mass conditions for the vectorial and spinorial weights read

$$\begin{aligned} \text{vec}: \quad & \frac{k_0}{2} + k_1\Big(\frac{1}{2} + \frac{\alpha_1}{4}\Big) = \frac{11}{16} - \tilde{N} - \frac{1}{2}(\tilde{v}_1^{\text{vec}})^2\,, \\ \text{spin}: \quad & k_1\Big(\frac{1}{2} - \frac{\alpha_1}{4}\Big) + \frac{k_2}{2} = \frac{11}{16} - \tilde{N} - \frac{1}{2}(\tilde{v}_1^{\text{spin}})^2\,, \end{aligned} \tag{4.22}$$

with

$$\tilde{N} = \frac{r_1 + r_2}{4} + \frac{r_3}{2}\,, \qquad \frac{1}{2}(\tilde{v}_1^{\text{vec}})^2 = \frac{1}{32}(n_1 + 4n_2)\,, \qquad \frac{1}{2}(\tilde{v}_1^{\text{spin}})^2 = \frac{1}{32}(n_1 + 4n_0)\,, \tag{4.23}$$

where we have used that $s_1 = s_2 = -$. The possible solutions to $\tilde{N} = 0, \frac{1}{4}$ and $\frac{1}{2}$ are given in table 7. The GSO projections determines the chirality of the spinors:

$$\text{vec}: \quad (-)^{k_0 + k_1}\,, \qquad \text{spin}: \quad (-)^{n_0 + k_0 + k_1}\,, \tag{4.24}$$

for the vectorial and spinorial weight representations, respectively. The GSO projection for the vectorial weights only depends on the representation of the twisted states, while for the spinorial weights it also depends on the model via n_0. In table 8 we have given the resulting spectrum for both vectorial and spinorial weights. We use $\alpha = (-)^{n_0}$ to denote the model dependent chirality of the spinorial weights. By combining the two tables 7 and 8 the four dimensional twisted states given in the last column of table 4 are obtained for the ten four dimensional model tabulated there.

repr. $\backslash$ m_1	2	6	10	14
$(\mathbf{1}, \mathbf{2}^{\mathbf{m_1-1}}_{\alpha_1})$	$\lvert a\rangle, \lvert \underline{a}\rangle$	$\lvert 0\rangle$		
$(\mathbf{2m_0}, \mathbf{2}^{\mathbf{m_1-1}}_{-\alpha_1})$	$\lvert 0\rangle$			
$(\mathbf{2}^{\mathbf{m_0-1}}_{-\alpha_0}, \mathbf{2m_1})$				$\lvert 0\rangle$
$(\mathbf{2}^{\mathbf{m_0-1}}_{\alpha_0}, \mathbf{1})$			$\lvert 0\rangle$	$\lvert a\rangle, \lvert \underline{a}\rangle$

Table 9: The states on the six dimensional hyper surfaces are determined by $m_0 = n_0 + n_2 + \cdots$ and $m_1 = n_2 + n_3 + \cdots$ from the $\mathbb{Z}_{2N}$ shift vector (4.1) using the conditions (4.28). The upper two rows correspond to vectorial weight representations and the lower two to spinorial weights. The holonomy doublet states are denoted by $|a\rangle$ and $|\underline{a}\rangle$ with $a = 1, 2$. The chiralities are expressed in terms of $\alpha_1 = (-)^{n_2+n_3+n_6+\cdots}$ and $\alpha_0 = (-)^{n_0+n_3+n_4+\cdots}$.

4.3.3 Six dimensional matter: the Nth twisted sector in $\mathbb{Z}_{2N}$ orbifolds

Our final example focuses the Nth twisted sector which is contained in all the $\mathbb{Z}_{2N}$ models. These states live on six dimensional tori T^2 and orbifolds $T^2/\mathbb{Z}_2$ located at the fixed points of $T^4/\mathbb{Z}_2$ embedded inside the orbifold $T^6/\mathbb{Z}_{2N}$. Starting from the shift (4.1) we find the shift vector of the Nth twisted sector

$$v_N^{\text{vec}} = \frac{1}{2}(0^{n_0}, 1^{n_1}, 0^{n_2}, \ldots)\,, \qquad v_N^{\text{spin}} = \frac{1}{2}(1^{n_0}, 0^{n_1}, 1^{n_2}, \ldots)\,, \tag{4.25}$$

using the integral shifts

$$d_N^{\text{vec}} = (0^{n_0+n_1}, 1^{n_2+n_3}, \ldots)\,, \qquad d_N^{\text{spin}} = d_N^{\text{vec}} - (1^{n_0}, 0^{n_1}, 1^{n_2}, 0^{n_3}, \ldots)\,. \tag{4.26}$$

Even though the signs of v_N^{vec} are all positive, its entries are not order. With the definitions $m_0 = n_0 + n_2 + \ldots$ and $m_1 = n_1 + n_3 + \ldots$, the ordered versions

$$\tilde{v}_N^{\text{vec}} = \frac{1}{2}(0^{m_0}, 1^{m_1})\,, \qquad \tilde{v}_N^{\text{spin}} = \frac{1}{2}(1^{m_0}, 0^{m_1})\,, \tag{4.27}$$

are obtained using the matrix S_N. (This matrix S_2 for the $\mathbb{Z}_4$ case is given in (4.20).) The mass level and GSO conditions take very simple forms which are readily solved

$$\begin{aligned} \text{vec}:\ \ m_1 &= 6 - 4k_0 - 8\tilde{N}\,, \quad \alpha_1 = (-)^{n_2+n_3+n_6+\cdots}\,, \\ \text{spin}:\ \ m_0 &= 6 - 4k_1 - 8\tilde{N}\,, \quad \alpha_0 = (-)^{n_0+n_3+n_4+\cdots}\,. \end{aligned} \tag{4.28}$$

Here α_0 and α_1 denote the chirality of the spinors for $k_1 = 0$ and $k_0 = 0$, respectively, which are clearly model depend. States that contain the vectors of the SO groups have spinors with the opposite chirality as compared to those states without vectors. The results of this analysis have been collected in table 9. The bosonic excitations with $\tilde{N} = 1/2$ generate the states $|a\rangle$ and $|\underline{a}\rangle$ with $a = 1, 2$ that form two doublets under the SU(2) holonomy group, and the vacuum $|0\rangle$ with $\tilde{N} = 0$ is a holonomy singlet. As there are only two different bosonic representations, we have chosen to collect the spectrum in a single table. We see that the six dimensional states (given in table 4) contained in the $T^6/\mathbb{Z}_4$ models discussed in section 3, in fact, provide the general structure of matter at the fixed points of $T^4/\mathbb{Z}_2$.

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4.4 The complete string spectrum in six and four dimensions

In the previous section we have explained an efficient method to compute the irreducible twisted states in heterotic SO(32) orbifold models. With that construction the spectra of these states were collected in a couple of tables for large classes of models simultaneously. In this subsection we use such tables to describe the full massless spectra of $\mathbb{Z}_{2N}$ models in a field theoretical language.

For completeness we begin with the gauge sector of the theory. In four dimensions the $\mathbb{Z}_{2N}$ orbifold conditions for the ten dimensional super Yang-Mills theory take the form

$$A_\mu^w \to e^{2\pi i\, v\cdot w}\, A_\mu^w\,, \tag{4.29}$$

where w are the SO(32) roots and the shift v is of the form (4.1). (The Cartan subalgebra gauge fields A_μ^I always survive the orbifolding.) The four dimensional gauge group therefore is determined by $v\cdot w \equiv 0$, its general form is given by (4.11). The untwisted sector matter is obtained from the orbifolding of the gauge fields with internal spacetime indices

$$A_i^w \to e^{2\pi i (w\cdot v - \phi_i)} A_i^w\,. \tag{4.30}$$

Their zero mode spectrum is obtained from the condition $w\cdot v - \phi_i \equiv 0$. In addition to these charged states, there may be neutral untwisted matter that arises from the ten dimensional supergravity theory.

At the six dimensional hyper surfaces the gauge group and untwisted spectrum is determined in an analogous way: One only has to replace $\phi \to N\phi$ and $v \to Nv$. In fact, the six dimensional gauge group is always $\mathrm{SO}(2m_0)\times\mathrm{SO}(2m_1)$ where $m_0 = n_0 + n_2 + \ldots$ and $m_1 = n_1 + n_3 + \ldots$. The common feature of the $\mathbb{Z}_{2N}$ models that we are focusing on is that the Nth twisted sector lives on the six dimensional hyper surfaces within the orbifold $T^6/\mathbb{Z}_{2N}$. In the previous subsection we have used string techniques to determine these six dimensional states as the twisted sector of $T^4/\mathbb{Z}_2$. Table 9 summarizes the resulting spectrum and may be used to derive the Nth twisted spectrum as follows: The value of m_1 decides which column is relevant for the spectrum. If the entry of the table is empty the representation on the left of the corresponding row is not part of the spectrum. If the entry is $|0\rangle$ the spectrum contains a holonomy singlet state in the representation determined by its row. Finally if the entry is $|a\rangle, |\underline{a}\rangle$ the corresponding representation forms two SU(2) holonomy doublets. Notice that all states in table 9 contain spinors. Their chirality is model dependent and determined by the signs α_0 and α_1 defined in the caption. If m_0 or m_1 is zero there cannot be a spinor, in that case the state only survives if the would-be chirality is positive. The spectrum of the six dimensional twisted matter given in table 4 has been determined this way. Being supersymmetric six dimensional matter, these states are hyper multiplets. The holonomy singlets form half-hyper multiplets, i.e. hyper multiplets that satisfy a reality condition. This completes the identification of the six dimensional matter in $T^6/\mathbb{Z}_{2N}$ orbifolds.

Also for the four dimensional twisted matter we have produced tables from which their spectrum can be read off for any specific $\mathbb{Z}_{2N}$ model. However, as these spectra can be rather involved, a combination of three tables have to be employed to identify the spectrum. Let us explain how one obtains spectra from such tables by the example of the first twisted sector of a $\mathbb{Z}_4$ orbifold discussed in subsection 4.3.2. In the two tables 8 the possible $\mathrm{SO}(2n_0) \times \mathrm{U}(n_1) \times \mathrm{SO}(2n_2)$ representations are given. They correspond to two different types of states in string theory. The chiralities of the spinors in the table on the left are fixed, while on the right they depend on value of n_0. Like for the Nth twisted sector described above, these states are part of the spectrum only if model dependent quantities ($n_1 + 4n_2$ and $n_1 + 4n_0$ in this case) take specific values given on the top rows of these tables, and only if the corresponding table entry is not empty. If filled, the value $\tilde{N}$ of the table entry determines the spacetime properties of the representation of the corresponding column via table 7: It fixes the internal space index structure of these states. (From a field theoretical point of view this is surprising because why should localized states have indices in space direction to which they cannot propagate.) This is an important information since it determines the multiplicities of states, or more precisely, their SU(2) holonomy properties. The $\mathbb{Z}_4$ models only have one four dimensional sector of chiral multiplets (as the third twisted sector is the conjugate of the first), but other $\mathbb{Z}_{2N}$ model may contain a number of them, as the appendices show.By providing such a sets of tables for each of them their spectra are fully specified.

For the irreducible twisted sectors of $\mathbb{Z}_{2N}$ these tables to together with transformation matrices S_p that indicate how to obtain them from the first twisted sector, completely specify their four dimensional spectrum. For reducible twisted sectors the situation is more complicated, as these sectors reside on fixed points of $\mathbb{Z}_M$ subgroups of the full orbifold group. These fixed points may correspond to four or six dimensional spacetime hyper surfaces. Independently of the dimension, there is a residual action of $\mathbb{Z}_{2N}$ on these fixed points, which can have a multitude of consequences on the spectrum of states on these hyper surfaces. Whether this leads to identifications, projections, or further orbifolding of these states depends on the geometrical action of the residual $\mathbb{Z}_{2N}$ action only. We now describe the various possibilities:

As observed at the beginning of this section the residual orbifold action can lead to the identification of sets of $\mathbb{Z}_M$ fixed points. Hence also the twisted matter living on these fixed points will be identified; no states are projected away. The residual orbifold action may break the gauge group further. This leads to branching of representations of the four dimensional twisted states with respect to the global gauge group. The chirality of the spectrum is not lost in this process for four dimensional states. If the identified fixed points correspond to six dimensional hyper surfaces, the resulting Kaluza-Klein spectrum in four dimensions is never chiral.

The other possibility is that the residual $\mathbb{Z}_{2N}$ action leaves the $\mathbb{Z}_M$ fixed point fixed as well. If the dimension of this fixed hyper surface is four, this leads to a projection of the spectrum at this fixed point; while if it is six, these states are orbifolded on $T^2/(\mathbb{Z}_{2N}/\mathbb{Z}_M)$. To describe how the spectrum is affected by this, we introduce some notation: A generic state with *vect*/*spin*orial weight $\tilde{w}$ and r_i internal space indices i (if $s_i = +$) or $\underline{i}$ (if $s_i = -$) in the pth twisted sector can be denoted by $|^{\tilde{w}}_{s,r}\rangle^{vec,spin}_p$. Note that even the untwisted matter can be represented like this: $A^w_i = |^w_i\rangle$. On such a state the $\mathbb{Z}_{2N}$ residual action takes the form

$$|^{\tilde{w}}_{s,r}\rangle_p \to e^{2\pi i\,\theta_p} \exp\left\{2\pi i\left(vS_p\cdot\tilde{w} - \sum_i s_i r_i \phi_i\right)\right\}|^{\tilde{w}}_{s,r}\rangle_p\,, \tag{4.31}$$

as follows from the orbifold condition in (4.12). The second phase factor can be understood easily in field theory: It is precisely the transformation property under the orbifold action of a state in a representation corresponding to weight $\tilde{w}$ that carries internal space indices parameterized by s and r. (The matrix S_p takes into account that in the classification of these states, and the resulting tables, $\tilde{w}$ refers to weights in the standard representation given in table 5.) The first phase factor

$$\theta_p = \frac{1}{2}(\phi^2 - v^2)p + vS_p\cdot\tilde{v}_p - \phi\cdot\tilde{\phi}_p \tag{4.32}$$

has a primarily stringy origin: It results from modular invariance and spectral flow, see (4.10). For six dimensional states (4.31) it dictates the orbifold boundary conditions, while for four dimensional states it defines a projection. In both cases the four dimensional zero mode spectrum that survives the orbifold projection satisfies

$$vS_p\cdot\tilde{w} - \sum_i \phi_i s_i r_i \equiv \frac{1}{2}(v^2 - \phi^2)p - vS_p\cdot\tilde{v}_p + \phi\cdot\tilde{\phi}_p\,. \tag{4.33}$$

This analysis does not take the four dimensional chirality into account, but this can be obtained straightforwardly from (4.7). However, if one wants to directly compare to the untwisted and first twisted spectrum when this chirality is negative, one needs to take the complex conjugate of the resulting spectrum. Using these steps the resulting four dimensional spectrum for any reducible twisted sector can be determined.

Before concluding this section, we would like to return to the $\mathbb{Z}_4$ example one final time to illustrate the orbifolding of six dimensional states. In the fourth column of table 4 the six dimensional spectrum is obtained from table 9. The four dimensional spectrum on the fixed points of $T^2/\mathbb{Z}_2$ has been determined from the orbifold projection (4.33). Since the chirality of the second twisted sector is negative according to (4.7), we have conjugated all representations so that table 4 compares only four dimensional states with the same (positive) chirality.

Let us close this section with a couple of final comments about the appendices B, C and D the $\mathbb{Z}_7$, $\mathbb{Z}_6$ and $\mathbb{Z}_8$ orbifold models. In these appendices we classify the modular invariant shifts and the irreducible twisted states only, since they together specify the full heterotic orbifold model. We refrain from computing the full four dimensional spectrum, as that can be obtained using the field theoretical techniques reviewed in this subsection. As explained with the example of the first twisted sector of $\mathbb{Z}_4$ orbifolds in subsection 4.3.2, we use collection of tables to specify all details of the irreducible twisted spectra. Moreover, since all irreducible twisted spectra can be obtained from the first twisted sector, we only give this spectrum and the matrices that give the other irreducible twisted states. Presented in this way the spectra of heterotic string models can be used in a variety of ways. In the next section we mention a few possible applications of our classification of orbifold models.

5. Applications and extensions

This section is devoted to some further extensions of our classifications and possible applications.

5.1 More general classes of orbifolds

The explicit classification of models in this work has been restricted to odd order orbifolds and $\mathbb{Z}_{2N}$ orbifolds with vectorial structure, and where only the Nth twisted sector is six dimensional. In particular, we have neglected the large class of $\mathbb{Z}_N \times \mathbb{Z}_{N'}$ models. Moreover, we have only focused on vectorial shift vectors. These omissions have been made for the sake of brevity, rather than as a matter of principle. In this subsection we take the opportunity to argue that our methods can be extended without any severe obstacle to include such models as well.

First of all, for even order orbifolds we may also consider spinorial shifts[1]

$$u = \frac{1}{4N}\big(1^{n_1}, 3^{n_2}, \ldots, (2N-1)^{n_N}\big)\,, \qquad \text{with} \sum_{k=1}^{N} n_k = 16\,, \tag{5.1}$$

which give rise to a product of $\mathrm{U}(n)$ gauge groups only

$$\mathrm{SO}(32) \to \mathrm{U}(n_1) \times \mathrm{U}(n_2) \times \cdots \times \mathrm{U}(n_N)\,. \tag{5.2}$$

The classification of models with vectorial and spinorial shift vectors looks different. The spinorial shifts (5.1) can be classified in much the same way as the vectorial ones in subsection 4.1. In this case the modular invariance condition gives the linear equation

$$N\,\phi^2 \equiv N\,u^2 = \frac{1}{16N}\sum_{k=1}^{N}(2k-1)^2\, n_k\,. \tag{5.3}$$

As this can be treated as a linear system for the numbers n_k, the same method of null-solutions may be applied as described in subsection 4.1.

Not only can the spinorial shift vectors be classified, also our method of systematically determining all twisted states applies with only a few minor modifications. The allowed representations in the irreducible twisted sectors are still determined by the mass conditions, (4.14) and (4.17), except that, since the resulting gauge group (5.2) never contain $\mathrm{SO}(2n)$ groups, there are no k_0 or k_m contributions. For the same reason the GSO conditions, (4.15) and (4.18), are now always true projections, rather than selection rules for the chirality of spinor representations.

Also the extension of our classification method to $\mathbb{Z}_N \times \mathbb{Z}_{N'}$ orbifolds is straightforward. This class of orbifolds also includes $\mathbb{Z}_{2N}$ orbifolds for which the Nth twisted sector is not a six dimensional sector on the orbifold $T^4/\mathbb{Z}_2$. A $\mathbb{Z}_N \times \mathbb{Z}_{N'}$ orbifold is defined by two

[1] For odd order orbifold models this simply corresponds to interchanging the spin-structures, i.e. the interchange of vectorial and spinorial weights.

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spacetime shifts, ϕ and ϕ', and two gauge shifts, v and v'. The requirements of modular invariance are [7, 19]

$$\frac{1}{2}N_{p\,p'}\Big(\big(p\phi+p'\phi'\big)^2-\big(pv+p'v'\big)^2\Big)\equiv 0\,, \tag{5.4}$$

for all p, p', where $N_{p\,p'}$ is the order of the shift $p\phi + p'\phi'$. Using similar techniques as employed in subsection 4.1, all solutions can be determined by some linear algebra. For a $\mathbb{Z}_N$ orbifold the signs of the shift vector can always be rotated away, but this need not be the case anymore because of the third condition in (5.4). The $(NN'-1)$ twisted sectors are labeled by two integers $p = 0, \ldots, N-1$ and $p' = 0, \ldots, N'-1$ not both equal to zero. To decide whether the (p, p') sector is four or six dimensional, one computes the relevant spacetime shift $p\phi + p'\phi'$. For each sector we may define the generalization of (4.10) by

$$\begin{aligned}\tilde{v}^{\mathrm{vec}}_{p\,p'} &= (pv+p'v'-d^{\mathrm{vec}}_{p\,p'})S_{p\,p'}\,,\\ \tilde{v}^{\mathrm{spin}}_{p\,p'} &= \left(pv+p'v'-\frac{1}{2}e-d^{\mathrm{spin}}_{p\,p'}\right)S_{p\,p'}\,,\end{aligned}\qquad \tilde{\phi}_{p\,p'}=p\phi+p'\phi'+\frac{1}{2}e_3+\delta_{p\,p'}\,. \tag{5.5}$$

Hence the spectrum classification formulas (4.14) and (4.17) and GSO projections (4.15) and (4.18) may be used to determine all irreducible twisted states in each (p, p') sector.

Finally, also the extension to models with Wilson lines can be performed without difficulty. As it is well-known also modular invariance put stringent conditions on the possible Wilson lines [7], which can be analyzed in a similar fashion as the multiple shift vectors for $\mathbb{Z}_N \times \mathbb{Z}_{N'}$ orbifolds. For the twisted spectrum the situation is fully identical to the one studied at length in this work, since, as we used in subsection 4.2, the twisted states are determined by the local shift vector only. In theories with Wilson lines not all fixed points are equivalent, but to each fixed point a local shift vector is associated which determines the complete (irreducible twisted) spectrum at this fixed point [52]. Hence, for models with Wilson lines our method of computing spectra is very efficient.

5.2 Applications to the $\mathrm{E}_8 \times \mathrm{E}_8$ theory

The heterotic $\mathrm{E}_8 \times \mathrm{E}_8{}'$ theory has been studied much in the past since this string theory was the first that looked promising for phenomenology. However, to go beyond identifying the gauge group and untwisted states always proved difficult because it was impossible to recognize patterns in the twisted states. One of the reasons for this is the appearance of various exceptional groups, whose representations do not follow easily identifiable patterns. The method that we have used to classify the twisted states of the heterotic SO(32) string on orbifolds in subsection 4.2 can be extended to the $\mathrm{E}_8 \times \mathrm{E}_8{}'$ theory, as we will now demonstrate. The central observation is to classify the subgroups and their representations of the maximal subgroup $\mathrm{SO}(16) \times \mathrm{SO}(16)'$ rather than those of the $\mathrm{E}_8 \times \mathrm{E}_8{}'$ group itself.

Using elementary representation theory which can be found in the tables of [53] the identification of representations of exceptional groups is not difficult. For example, as is well-known, the group E_8 can be understood as the spinor bundle over SO(16). This exemplifies that the representations of E_8 and other subgroups can be easily understood as combinations of representations of (maximal) regular subgroups.

In an $\mathrm{SO}(16)\times\mathrm{SO}(16)'$ Cartan basis a gauge shift is now described by the combination (v,v') of two shift vectors. In general each of the SO(16) groups is broken to

$$\mathrm{SO}(16)\to\mathrm{SO}(2n_0)\times\mathrm{U}(n_1)\times\cdots\times\mathrm{U}(n_{N-1})\times\mathrm{SO}(2n_N)\,. \tag{5.6}$$

We may denote the $\mathrm{SO}(16)\times\mathrm{SO}(16)'$ weights as (w,w'), which can independently be vectorial or spinorial weights. By using the definitions (4.10) we bring the weights in the standard form of table 5, and therefore the states can be classified as before. In particular, the mass formulas are now split into four sectors:

$$\begin{aligned}
(\mathrm{vec},\mathrm{vec}') :\;\; N_{\mathrm{vec}}+N'_{\mathrm{vec}} &= \frac{5}{8}+\frac{1}{2}\tilde{\phi}_p^2-\tilde{N}\,,\\
(\mathrm{vec},\mathrm{spin}') :\;\; N_{\mathrm{vec}}+N'_{\mathrm{spin}} &= \frac{5}{8}+\frac{1}{2}\tilde{\phi}_p^2-\tilde{N}\,,\\
(\mathrm{spin},\mathrm{vec}') :\;\; N_{\mathrm{spin}}+N'_{\mathrm{vec}} &= \frac{5}{8}+\frac{1}{2}\tilde{\phi}_p^2-\tilde{N}\,,\\
(\mathrm{spin},\mathrm{spin}') :\;\; N_{\mathrm{spin}}+N'_{\mathrm{spin}} &= \frac{5}{8}+\frac{1}{2}\tilde{\phi}_p^2-\tilde{N}\,,
\end{aligned} \tag{5.7}$$

where we have used the shorthand notations

$$\begin{aligned}
N_{\mathrm{vec}} &= \frac{k_0}{2}+\sum_{a=1}^{m-1}k_a\left(\frac{1}{2}+\alpha_a\tilde{v}_{p\,a}^{\mathrm{vec}}\right)+\frac{1}{2}(\tilde{v}_p^{\mathrm{vec}})^2\,,\\
N_{\mathrm{spin}} &= \frac{k_m}{2}+\sum_{a=1}^{m-1}k_a\left(\frac{1}{2}+\alpha_k\tilde{v}_{p\,a}^{\mathrm{spin}}\right)+\frac{1}{2}(\tilde{v}_p^{\mathrm{spin}})^2\,,
\end{aligned} \tag{5.8}$$

for both SO(16) and SO(16)$'$ weights. The solutions to the mass relations (5.7) can be solved as quickly as the ones given in (4.14) and (4.17) of the SO(32) theory. On both SO(16) weights the GSO projections, (4.15) and (4.18), are applied.

As our discussion here showed, our classification method can be applied directly to the $\mathrm{E}_8\times\mathrm{E}_8{}'$ theory as well, and will also in that case give fast classification results. More importantly, it will also make the patterns in the twisted spectra transparent.

5.3 Heterotic/type-I duality on odd order orbifolds

The classification procedure that we describe in this paper might be useful in the context of the S-duality [30] between the heterotic SO(32) string and type-I models. We illustrate this by make some comments on the present status of this duality in four dimensions. We begin by briefly recalling the construction of type-I models. The starting point is type-II closed string theory with an operator Ω that reverse the spatial worldsheet coordinate. The projection on Ω-invariant states induces the presence of non-dynamical orientifold planes, that are sources for RR-flux. This can be seen by computing the closed-string partition function in presence of an Ω-projection: As is well known, both the standard torus and a Klein bottle amplitude are present. The latter can be seen as a tree-level exchange of closed strings between two sources, i.e. the orientifold planes. If the internal space is compact, consistency of this construction is enforced by requiring that all these tadpoles are canceled which requires specific sets of D-branes [22]. (In a rather different but earlier approach the construction was also described in [21]). A stack of D-branes gives rise to $\mathrm{U}(n)$, $\mathrm{Sp}(2n)$ or $\mathrm{SO}(2n)$ gauge groups generated by their Chan-Paton labels [54] (see also [55]). The tadpole cancellation conditions is powerful enough to ensure that irreducible anomalies do not arise in any dimension [56]. In particular in ten dimensions the type-I theory is required to be an $\mathrm{SO}(32)$ gauge theory due to the 32 $D9$-branes.

In ten dimensions the resulting type-I supergravity is described by the same action as the heterotic $\mathrm{SO}(32)$ supergravity but with a different coupling. On this observation the strong/weak duality between these two theories on the supergravity level was based, and then extended as a string duality in ten dimensions in [30]. Ref. [57, 58] consider the duality relation between the dilatons of the two theories in various dimensions by toroidal compactification. In particular, these authors observe that in four dimension the duality is a weak/weak duality in the supergravity approximation. For more complicated compactification, like orbifolds, the status of the duality is less clear. In particular, the consistency requirement of modular invariance of the heterotic string, which enforces the existence of twisted states, does not have a counterpart for the open string in type-I. In this subsection we investigate the four dimensional version of the heterotic/type-I S-duality by matching gauge group and chiral spectra of four dimensional heterotic $\mathrm{SO}(32)$ orbifold models to those of type-I orbifolds.[2]

The details of the classification on the type-I side was established by [57, 58], where also previous results were collected, we refer to these papers for a detailed list of references. Concretely, the tadpole cancellation conditions impose that

$$\mathrm{Tr}\big[\gamma^{2k}\big] = 32 \prod_{i=1}^{3} \cos(\pi k \phi_i), \qquad \text{with } \gamma = e^{-2i\pi v\cdot H}\,, \tag{5.9}$$

for $0 \leq k \leq N-1$, where the Cartan generators H_I act on the $\mathrm{SO}(32)$ Chan-Paton factors. These conditions fix the shift v uniquely. In particular, the $\mathbb{Z}_3$ orbifold of type-I theory has $v = (0^4, 1^{12})/3$, which gives the same gauge group and charged untwisted matter fields as in the heterotic $\mathbb{Z}_3$ model with $n = 4$, see section 2. Only the untwisted states give rise to a charged spectrum in type-I models, since these models cannot contain charged twisted states. This gives a partial explanation why there is one $\mathbb{Z}_3$ type-I model: As can be seen from (2.5) only for $n = 4$ the dual heterotic theory does not have irreducible anomalies in its untwisted spectrum. But anomaly cancellation does not provide a complete answer, since on the heterotic side also the $n = 0$ model is obviously anomaly free, but it has no type-I counterpart. The situation is identical for the other odd orbifold $\mathbb{Z}_7$: From the untwisted spectrum of the heterotic $\mathbb{Z}_7$ models given in appendix B it follows that two model are free of irreducible anomalies, see (B.7). One has the shift vector $v = (0^4, 1^4, 2^4, 3^4)/7$; for this shift a type-I model exists, see [58], while the trivial embedding ($v = 0$) again has no type-I dual.

There are more aspects that need to be addressed to establish how the S-duality is a true duality between type-I and heterotic theories in four dimensions. First of all, also the

[2] We refer to [59, 60] for a study of this S-duality of the corresponding low-energy four dimensional supergravity theories.

other heterotic $\mathbb{Z}_3$ models should have duals on the type-I side. Such dual models require additional four dimensional states to appear in order to cancel the irreducible anomaly. Maybe these 'twisted open string states' can be understood as D-string excitations [61].

Moreover, even for type-I and heterotic models with equivalent $\mathbb{Z}_3$ and $\mathbb{Z}_7$ shifts, the spectra of these dual theories are not equal. It has been argued in [62, 63] the $\mathbb{Z}_3$ twisted states become massive when the orbifold singularities are blown up, and then the remaining massless charged spectra match. Another profound difference between these two models is that the cancellation of the leftover reducible anomalies is fundamentally different for heterotic and type-I models. In heterotic models the Green-Schwarz mechanism always descents from the ten dimensional one, and therefore requires a unique factorization. As we saw in section 2 for this factorization the charged twisted states are essential, see (2.8) for example. Instead, in type-I orbifold models, it is sufficient for local anomaly cancellation that the anomaly polynomial factorizes, as the anomalous couplings of the neutral twisted closed particles at the fixed points are sufficiently flexible [64, 65, 66].

In the discussion above we have primarily focused on the two odd order orbifolds, therefore one may wonder what the status is of the heterotic/type-I duality on even order orbifolds. As was first observed in ref. [23], even order orbifolds require that both $D9$- and $D5$-branes are introduced. This system has both gauge groups in ten and six dimensions. This construction has been extended to four dimensions by various groups [67]–[70]. In particular, the authors of ref. [58] showed that there are no $\mathbb{Z}_4$ type-I models, that fulfill the tadpole cancellation conditions. As we have seen in section 3, there exist ten perturbative heterotic $\mathbb{Z}_4$ models, but there are apparently no possible type-I duals. The gauge groups localized on the six dimensional $D5$-branes do not have perturbative counter parts on the heterotic side, but might be related to non-perturbative $M5$-brane excitations. The $M5$-branes give rise to non-modular invariant heterotic models. Part of their spectrum can sometimes be matched to that of type-I models [50]. The duality between heterotic and type-I four dimensional models in general, and for even order orbifolds in particular, still requires further research. We hope that our classification of heterotic SO(32) models may provide a useful testing ground for new proposals for more precise definitions of this duality.

5.4 Model searches

As a final application we mention that our classification procedure can be very useful for string model searches. This can be searches for MSSM-like, GUT or orbifold GUT models. Irrespectively of which kind of model one is looking for, the basic strategy is the same: First find a model with the appropriate gauge group, secondly check whether at least the wanted matter representations are present, next fill in the details of the spectrum. After this, more detailed investigations can be undertaken in which the forms of (perturbative) superpotentials, gauge kinetic terms and Kähler terms are obtained.

Our methods can clearly be aimed at tackling the first part of this program, as it only relies on the spectrum of the theory. Given the gauge group one is looking for, one can immediately select possible orbifolds models having such a gauge group unbroken. Since we have made the connection between the resulting gauge groups and the gauge shifts explicit in (4.1) and (4.2), we can identify the appropriate values of some of the integers n_i in the gauge shift. The requirement of modular invariance then quickly tells us if solutions can be found at all. Using our tables one has immediate overview of the spectrum of the theory.

Let us close this section on applications by illustrating how model searches can be performed using our classification. Suppose one is interested in obtaining SO(10)-like GUTs from string theory. For the even order $\mathbb{Z}_{2N}$ orbifolds, SO(10) groups arise if $n_0 = 5$ or $n_N = 5$. Moreover, since we have shown that the six and four dimensional twisted sectors contain spinor representations, it is quite likely that the chiral string spectrum contains spinors of SO(10). In particular if we focus on the $\mathbb{Z}_4$ model, it follows from the modular invariance condition that $n_1 = 2, 10$ for either $n_0 = 5$ or $n_2 = 5$. These solutions correspond precisely to the two models found in section 3 from table 4. (the other two are the same as these ones as they are obtained by interchanging the roles of n_0 and n_2.)

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A. Non-compact holomorphic partition function

In this appendix we collect some properties of the heterotic SO(32) string partition function on which the analysis of the main text relies heavily. The partition function is constructed out of the following modular forms:

$$\vartheta \left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](\tau) = e^{2\pi i\alpha\beta}\, q^{\frac{1}{2}\alpha^2} \prod_{n\geq 1}\Big\{(1-q^n)\prod_{s=\pm}(1+e^{-2\pi i s\beta}\, q^{n-\frac{1}{2}-s\alpha})\Big\}, \quad \eta(\tau) = q^{\frac{1}{24}}\prod_{n\geq 1}(1-q^n)\,, \tag{A.1}$$

with $q = e^{2\pi i\,\tau}$. In the sum representation the theta function can be easily generalized to vector valued characteristics. In particular for the heterotic SO(32) theory with spin-structures t, t' and p, p' at one loop we use the theta function

$$\vartheta \left[\begin{matrix}\frac{1-t}{2}\,e - p\,v\\ \frac{1-t'}{2}e - p'v\end{matrix}\right](\tau) = \sum_{n\in\mathbb{Z}^{16}} q^{\frac{1}{2}(n+\frac{t-1}{2}e+pv)^2}\, e^{2\pi i(\frac{t'-1}{2}e+p'v)\cdot(n+\frac{t-1}{2}e+pv)}\,, \tag{A.2}$$

such that the full holomorphic partition function can be written as

$$Z = \frac{1}{2N}\sum \tilde{\eta}^{p,t}_{p',t'}\,\frac{1}{\eta^{15}(\tau)}\vartheta\left[\begin{matrix}\frac{1-t}{2}\,e - p\,v\\ \frac{1-t'}{2}e - p'v\end{matrix}\right](\tau)\Big(\prod_i \vartheta\left[\begin{matrix}\frac{1}{2}+p\,\phi_i\\ \frac{1}{2}+p'\phi_i\end{matrix}\right](\tau)\Big)^{-1}\,, \tag{A.3}$$

where N is the order of the orbifold. The phases

$$\tilde{\eta}^{p,t}_{p',t'} = \exp 2\pi i \left(\frac{1}{2}(\phi^2 - v^2)\, pp' + p\frac{1-t'}{2} e \cdot v \right) \tag{A.4}$$

are determined by the requirement of modular invariance. This requirement also implies that

$$\frac{N}{2}(v^2 - \phi^2) \equiv 0\,. \tag{A.5}$$

The zero mode spectrum of the theory is determined by expanding the partition function (A.3) to the constant part in q. This expansion can be facilitated by first determining the minimal power that can arise from the theta function is the numerator and denumerator by using the periodicity of the theta functions in their upper characteristic. The product representation

$$\vartheta \begin{bmatrix} \frac{1}{2}+p\,\phi_i \\ \frac{1}{2}+p'\phi_i \end{bmatrix} = e^{2\pi i \tilde{\phi}_{p\,i}(p'\phi_i+\frac{1}{2})} q^{\frac{1}{2}\tilde{\phi}^2_{p\,i}} \prod_{m_i \geq 1} \Big\{ (1-q^{m_i}) \prod_{s_i = \pm} \Big(1 + e^{-2\pi i s_i (p'\phi_i + \frac{1}{2})} q^{m_i - \frac{1}{2} - s_i \tilde{\phi}_{p\,i}} \Big) \Big\} \tag{A.6}$$

is convenient when expanding $\vartheta \begin{bmatrix} \frac{1}{2}+p\,\phi_i \\ \frac{1}{2}+p'\phi_i \end{bmatrix}^{-1}$: since all terms with $m_i > 1$ give massive string states and can be ignored since we are only interested in the massless spectrum. Here we have defined the vector $\delta_p \in \mathbb{Z}^3$ such that all entries $\tilde{\phi}_p = p\phi + \frac{1}{2}e + \delta_p$, lie between $-\frac{1}{2} < \tilde{\phi}_{p\,i} \leq \frac{1}{2}$. We rewrite (A.2) for $t = 0$ as

$$\vartheta \begin{bmatrix} \frac{1}{2}\,e - p\,v \\ \frac{1-t'}{2} e - p' v \end{bmatrix} = \sum_{w \in \mathbb{Z}^{16}} q^{\frac{1}{2}(w + v_p^{\text{spin}})^2} e^{2\pi i (\frac{t'-1}{2} + p'v)(w + v_p^{\text{spin}})}\,, \tag{A.7}$$

where $d_p^{\text{spin}} \in \mathbb{Z}^{16}$ is chosen such that all entries $v_p^{\text{spin}} = pv - \frac{1}{2}e - d_p^{\text{spin}}$, lie between $-1/2 < v_{p\,I}^{\text{spin}} \leq 1/2$. Similarly for the vectorial weights ($t = 1$) we have

$$\vartheta \begin{bmatrix} -p\,v \\ \frac{1-t'}{2} e - p' v \end{bmatrix} = \sum_{w \in \mathbb{Z}^{16}} q^{\frac{1}{2}(w + v_p^{\text{vec}})^2} e^{2\pi i (\frac{t'-1}{2} e + p'v)(w + v_p^{\text{vec}})}\,, \tag{A.8}$$

with $d_p^{\text{vec}} \in \mathbb{Z}^{16}$ also chosen such that the entries of $v_p^{\text{vec}} = pv - d_p^{\text{vec}}$ lie in the same interval. The zero mode mass spectra are determined by the relations

$$\frac{1}{2}(w + v_p)^2 - \frac{5}{8} - \frac{1}{2}\tilde{\phi}_p^2 + \sum_i \left(\frac{1}{2} - s_i \tilde{\phi}_{p\,i} \right) r_i = 0\,, \tag{A.9}$$

with $s_i = \pm$ and integers $r_i \geq 0$ and $v_p = v_p^{\text{vec}}, v_p^{\text{spin}}$ for vectorial and spinorial weights, respectively. Moreover it can be recognized that the sums over t' and p' lead to GSO and orbifold projections that lead to the conditions

$$\begin{aligned} \text{GSO}: &\quad \frac{1}{2} e \cdot (w - d_p) \equiv 0\,, \\ \text{Orbifold}: &\quad \frac{1}{2}(\phi^2 - v^2)p + v \cdot (w + v_p) - \sum_i \phi_i (\tilde{\phi}_{p\,i} + s_i r_i) \equiv 0\,, \end{aligned} \tag{A.10}$$

on the spectrum for both the vectorial and spinorial weights.

$\mathbb{Z}_7$	$\phi = \frac{1}{7}(1,2,-3)$
$\phi_1 = \frac{1}{7}$	$[\overline{\mathbf{n_3}}]_2, (\mathbf{2n_0}, \mathbf{n_1}), (\overline{\mathbf{n_1}}, \mathbf{n_2}), (\overline{\mathbf{n_2}}, \mathbf{n_3})$
$\phi_2 = \frac{2}{7}$	$[\mathbf{n_1}]_2, (\mathbf{2n_0}, \mathbf{n_2}), (\overline{\mathbf{n_1}}, \mathbf{n_3}), (\overline{\mathbf{n_2}}, \overline{\mathbf{n_3}})$
$\phi_3 = -\frac{3}{7}$	$[\mathbf{n_2}]_2, (\mathbf{2n_0}, \overline{\mathbf{n_3}}), (\overline{\mathbf{n_1}}, \overline{\mathbf{n_2}}), (\mathbf{n_1}, \mathbf{n_3})$

Table 10: This table gives untwisted spectrum of the $\mathbb{Z}_7$ orbifold models.

$\tilde{N}$	0	$\frac{1}{7}$	$\frac{2}{7}$	$\frac{3}{7}$	$\frac{4}{7}$
states	$\vert 0\rangle$	$\vert\underline{1}\rangle$	$\vert\underline{1}^2\rangle, \vert\underline{2}\rangle$	$\vert\underline{1}^3\rangle, \vert\underline{12}\rangle, \vert\underline{3}\rangle$	$\vert\underline{1}^4\rangle, \vert\underline{1}^2\underline{2}\rangle, \vert\underline{2}^2\rangle, \vert\underline{13}\rangle, \vert\underline{3}\rangle$
multi.	1	1	2	3	5

Table 11: The possible values of $\tilde{N}$ for the first twisted sector of a $\mathbb{Z}_7$ are $0, \frac{1}{7}, \dots \frac{4}{7}$. The index structure of the corresponding states and total multiplicities are indicated.

vectorial repr.						
even $\quad\backslash\quad n_1+4n_2+9n_3$	odd $\quad\backslash\quad n_1+4n_2+9n_3+7$	14	28	42	56	70
$(\mathbf{1})$	$(\overline{\mathbf{n_3}})$	$\frac{4}{7}$	$\frac{3}{7}$	$\frac{2}{7}$	$\frac{1}{7}$	0
$([\overline{\mathbf{n_3}}]_2)$	$(\overline{\mathbf{n_2}}),([\overline{\mathbf{n_3}}]_3)$	$\frac{3}{7}$	$\frac{2}{7}$	$\frac{1}{7}$	0	
$(\overline{\mathbf{n_2}}, \overline{\mathbf{n_3}}),([\overline{\mathbf{n_3}}]_4)$	$(\overline{\mathbf{n_1}}),(\overline{\mathbf{n_2}}, [\overline{\mathbf{n_3}}]_2)$	$\frac{2}{7}$	$\frac{1}{7}$	0		
$(\overline{\mathbf{n_1}}, \overline{\mathbf{n_3}}),\ (\overline{\mathbf{n_2}}, [\overline{\mathbf{n_3}}]_3)$	$(\mathbf{2n_0}),\ (\overline{\mathbf{n_1}}, [\overline{\mathbf{n_3}}]_2),\ ([\overline{\mathbf{n_2}}]_2, \overline{\mathbf{n_3}})$	$\frac{1}{7}$	0			
$(\mathbf{2n_0}, \overline{\mathbf{n_3}}),\ (\overline{\mathbf{n_1}}, \overline{\mathbf{n_2}})$		0				$\tilde{N}$

spinorial repr. $\quad\backslash\quad \frac{1}{4}(49n_0+25n_1+9n_2+n_3)$	14	28	42	56	70
$(\mathbf{2}_{\alpha}^{\mathbf{n_0-1}})$	$\frac{4}{7}$	$\frac{3}{7}$	$\frac{2}{7}$	$\frac{1}{7}$	0
$(\mathbf{2}_{-\alpha}^{\mathbf{n_0-1}}, \mathbf{n_1})$	$\frac{3}{7}$	$\frac{2}{7}$	$\frac{1}{7}$	0	
$(\mathbf{2}_{-\alpha}^{\mathbf{n_0-1}}, \mathbf{n_2}),\ (\mathbf{2}_{\alpha}^{\mathbf{n_0-1}}, [\mathbf{n_1}]_2)$	$\frac{2}{7}$	$\frac{1}{7}$	0		
$(\mathbf{2}_{-\alpha}^{\mathbf{n_0-1}}, \mathbf{n_3}),\ (\mathbf{2}_{\alpha}^{\mathbf{n_0-1}}, \mathbf{n_1}, \mathbf{n_2}),\ (\mathbf{2}_{-\alpha}^{\mathbf{n_0-1}}, [\mathbf{n_1}]_3)$	$\frac{1}{7}$	0			
$(\mathbf{2}_{-\alpha}^{\mathbf{n_0-1}}, \overline{\mathbf{n_3}}),\ (\mathbf{2}_{\alpha}^{\mathbf{n_0-1}}, \mathbf{n_1}, \mathbf{n_3}),\ (\mathbf{2}_{\alpha}^{\mathbf{n_0-1}}, [\mathbf{n_2}]_2)$	0				$\tilde{N}$

Table 12: For an even (odd) shift v_{even} (v_{odd}) the first twisted spectrum has even (odd) vectorial weight representation. The chirality of the spinors in the spinorial weights is given by $\alpha = (-)^{n_0}$ ($\alpha = (-)^{n_0+1}$) for an even (odd) shift.

B. $\mathbb{Z}_7$ models

This appendix is devoted the $\mathbb{Z}_7$ models with spacetime shift $\phi = \frac{1}{7}(1,2,-3)$. The tables 10–12 on the next page give their complete spectrum. By Weyl reflections and additions of roots we can bring the gauge shift v in the standard forms:

$$v_{\text{even}} = \frac{1}{7}\big(0^{n_0}, 1^{n_1}, 2^{n_2}, 3^{n_3}\big)\,, \qquad v_{\text{odd}} = \frac{1}{7}\big(0^{n_0}, 1^{n_1}, 2^{n_2}, 3^{n_3-1}, -4\big)\,. \tag{B.1}$$

Notice that $v_{\text{odd}} = v_{\text{even}} - d_1$ with $d_1 = (0^{15}, 1)$. These shifts are not equivalent since for the modular invariance condition $3/7$ and $-4/7$ are not equivalent: For the shifts v_{even} the modular invariance condition is satisfied by null-solutions, that have the form

$$\nu = (14p_1, 7p_2, 14p_3) + q_1(4,-1,0) + q_2(9,0,-1)\,, \tag{B.2}$$

while the shifts v_{odd} are obtained by adding the shift $(3,1,0)$ to the null-solutions. Both shifts gives rise to the gauge group:

$$\mathrm{SO}(2n_0) \times \mathrm{U}(n_1) \times \mathrm{U}(n_2) \times \mathrm{U}(n_3)\,. \tag{B.3}$$

In table 10 we have given the untwisted matter spectrum.

The twisted matter is given by three irreducible sectors: The first, second and fourth twisted sector all carry the same four dimensional chirality (4.7). The vectorial weights do not contain spinor representations and therefore the GSO condition acts as a projection. For these representations in the first twisted sector this projection depends on which shift is used:

$$\text{First twisted}: \quad v_{\text{even}}: \ \frac{1}{2}\sum_{a=0}^{3} k_a \equiv 0\,, \qquad v_{\text{odd}}: \ \frac{1}{2}\sum_{a=0}^{3} k_a \equiv \frac{1}{2}\,. \tag{B.4}$$

We call the corresponding vectorial weights even and odd, respectively. The spinorial weights always contain a spinor representation and hence the GSO selects as usual only the chirality of this spinor: For even shifts we have $\alpha_{\text{even}} = (-)^{n_0}$, while for odd $\alpha_{\text{odd}} = (-)^{n_0+1}$. The content of the first twisted sector is summarized in tables 11 and 12.

The second twisted sector content is obtained simply by these tables by employing the transformation matrix

$$S_2 = \left(\begin{array}{c|c|c|c} \mathbb{1}_{n_0} & & & \\ \hline & & \mathbb{1}_{n_1} & \\ \hline & & & -\mathbb{1}_{n_2} \\ \hline & -\mathbb{1}_{n_3} & & \end{array}\right)\,, \tag{B.5}$$

and results in the replacements: $\mathbf{n_1} \to \overline{\mathbf{n_3}} \to \mathbf{n_2} \to \mathbf{n_1}$. However, the GSO condition for the vectorial weights is different: The vectorial second twisted sector is even (odd) if n_2+n_3 is even (odd). For the spinorial weights the chirality are selected by $\alpha = (-)^{n_0}$ for both even and odd shifts. The bosonic excitations are interchanged in table 11 as: $1 \to 3 \to 2 \to 1$. The transformation matrix

$$S_4 = (S_2)^2 = \left(\begin{array}{c|c|c|c} \mathbb{1}_{n_0} & & & \\ \hline & & & -\mathbb{1}_{n_1} \\ \hline & \mathbb{1}_{n_2} & & \\ \hline & & -\mathbb{1}_{n_3} & \end{array}\right)\,. \tag{B.6}$$

indicates that the fourth twisted sector is obtained from the first twisted sector by $\mathbf{n_1} \to \mathbf{n_2} \to \overline{\mathbf{n_3}} \to \mathbf{n_1}$. In this case the even (odd) vectorial weights are required for even (odd) n_1+n_2. The chirality of the spinorial weights reads $\alpha = (-)^{n_2+n_2+n_3}$. The interchange of the bosonic excitations in table 11 reads: $1 \to 2 \to 3 \to 1$.

In subsection 5.3 we consider the type-I/heterotic duality. To facilitate that discussion we give the irreducible anomalies of the untwisted states, which are listed in table 10. The corresponding anomaly polynomial reads

$$\begin{aligned} I_{6|u\ irr} = & -\frac{1}{6}\big(4-2n_0-n_1+2n_2\big)\,\mathrm{tr}\,F_1^3 - \frac{1}{6}\big(4-2n_0+n_1-2n_2+2n_3\big)\,\mathrm{tr}\,F_2^3 - \\ & -\frac{1}{6}\big(-4+2n_0-2n_1+n_3\big)\,\mathrm{tr}\,F_3^3 \end{aligned} \tag{B.7}$$

$\mathbb{Z}_6$-I	$\phi = \frac{1}{6}(1,1,-2)$
$\phi_{1,2} = \frac{1}{6}$	$(\mathbf{2n_0},\mathbf{n_1}), (\overline{\mathbf{n_2}},\mathbf{2n_3}), (\overline{\mathbf{n_1}},\mathbf{n_2})$
$\phi_3 = -\frac{1}{3}$	$[\overline{\mathbf{n_1}}]_2, [\mathbf{n_2}]_2, (\mathbf{n_1},\mathbf{2n_3}), (\mathbf{2n_0},\overline{\mathbf{n_2}})$

Table 13: This table gives the untwisted spectrum of the $\mathbb{Z}_6$ orbifold models.

$\tilde{N}$	0	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$
states	$\lvert 0\rangle$	$\lvert\underline{1}^k\underline{2}^{1-k}\rangle$	$\lvert\underline{1}^k\underline{2}^{2-k}\rangle, \lvert 3\rangle$	$\lvert\underline{1}^k\underline{2}^{3-k}\rangle, \lvert\underline{1}^k\underline{2}^{1-k}3\rangle$	$\lvert\underline{1}^k\underline{2}^{4-k}\rangle, \lvert\underline{1}^k\underline{2}^{2-k}3\rangle, \lvert\underline{3}\rangle, \lvert 3^2\rangle$
SU(2) hol.	$\mathbf{1}$	$\overline{\mathbf{2}}$	$\overline{\mathbf{3}},\mathbf{1}$	$\overline{\mathbf{4}},\overline{\mathbf{2}}$	$\overline{\mathbf{5}},\overline{\mathbf{3}},\mathbf{1},\mathbf{1}$
multi.	1	2	4	6	10

Table 14: The possible values of $\tilde{N}$ for the first twisted sector of a $\mathbb{Z}_6$-I are $0, \frac{1}{6}, \dots \frac{2}{3}$. The index structure of the corresponding states, their SU(2) holonomy representations and total multiplicities are indicated.

vectorial repr. \ $n_1 + 4n_2 + 9n_3$	6	18	30	42	54
$(\mathbf{2}_{+}^{\mathbf{n_3-1}})$	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$	0
$(\overline{\mathbf{n_2}},\mathbf{2}_{-}^{\mathbf{n_3-1}})$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$	0	
$(\overline{\mathbf{n_1}},\mathbf{2}_{-}^{\mathbf{n_3-1}})$, $([\overline{\mathbf{n_2}}]_2,\mathbf{2}_{+}^{\mathbf{n_3-1}})$	$\frac{1}{3}$	$\frac{1}{6}$	0		
$(\mathbf{2n_0},\mathbf{2}_{-}^{\mathbf{n_3-1}})$, $(\overline{\mathbf{n_1}},\overline{\mathbf{n_2}},\mathbf{2}_{+}^{\mathbf{n_3-1}})$, $([\overline{\mathbf{n_2}}]_3,\mathbf{2}_{-}^{\mathbf{n_3-1}})$	$\frac{1}{6}$	0			
$([\overline{\mathbf{n_1}}]_2,\mathbf{2}_{+}^{\mathbf{n_3-1}})$, $(\mathbf{2n_0},\overline{\mathbf{n_2}},\mathbf{2}_{+}^{\mathbf{n_3-1}})$, $(\mathbf{n_1},\mathbf{2}_{-}^{\mathbf{n_3-1}})$	0				$\tilde{N}$

spinorial repr. \ $n_2 + 4n_1 + 9n_0$	6	18	30	42	54
$(\mathbf{2}_{\alpha}^{\mathbf{n_0-1}})$	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$	0
$(\mathbf{2}_{-\alpha}^{\mathbf{n_0-1}},\mathbf{n_1})$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$	0	
$(\mathbf{2}_{-\alpha}^{\mathbf{n_0-1}},\mathbf{n_2})$, $(\mathbf{2}_{\alpha}^{\mathbf{n_0-1}},[\mathbf{n_1}]_2)$	$\frac{1}{3}$	$\frac{1}{6}$	0		
$(\mathbf{2}_{-\alpha}^{\mathbf{n_0-1}},\mathbf{2n_3})$, $(\mathbf{2}_{\alpha}^{\mathbf{n_0-1}},\mathbf{n_1},\mathbf{n_2})$, $(\mathbf{2}_{-\alpha}^{\mathbf{n_0-1}},[\mathbf{n_1}]_3)$	$\frac{1}{6}$	0			
$(\mathbf{2}_{\alpha}^{\mathbf{n_0-1}},[\mathbf{n_2}]_2)$, $(\mathbf{2}_{\alpha}^{\mathbf{n_0-1}},\mathbf{n_1},\mathbf{2n_3})$, $(\mathbf{2}_{-\alpha}^{\mathbf{n_0-1}},\overline{\mathbf{n_2}})$	0				$\tilde{N}$

Table 15: The full spectrum of the first twisted sector of $\mathbb{Z}_6$-I orbifold models can be read off from this table. To determine the appropriate multiplicities it should be combined with table above. The model dependent chirality reads $\alpha = (-)^{n_0}$. When n_0 or n_3 is zero, only those states should be kept with positive chirality.

where F_a are the U(n_a) field strengths. It is not difficult to show that with the constraint $n_0+n_1+n_2+n_3 = 16$, there is only a unique solution for which these irreducible anomalies are absent: $n_0 = n_1 = n_2 = n_3 = 4$.

C. $\mathbb{Z}_6$ models

The $\mathbb{Z}_6$-I models have the space shift $\phi = \frac{1}{6}\big(1,1,-2\big)$. Their full spectra are listed in tables 13–15 on the next page. The gauge shift and the level matching condition read for the $\mathbb{Z}_6$ theory

$$v = \frac{1}{6}\big(0^{n_0}, 1^{n_1}, 2^{n_2}, 3^{n_3}\big), \qquad 3v^2 = \frac{1}{12}n_1 + \frac{1}{3}n_2 + \frac{3}{4}n_3 \equiv \frac{1}{2}\,. \tag{C.1}$$

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$\mathbb{Z}_8$-I	$\phi = \frac{1}{8}(1,2,-3)$
$\phi_1 = \frac{1}{8}$	$(\mathbf{2n_0}, \mathbf{n_1}), (\overline{\mathbf{n_3}}, \mathbf{2n_4}), (\overline{\mathbf{n_1}}, \mathbf{n_2},), (\overline{\mathbf{n_2}}, \mathbf{n_3})$
$\phi_2 = \frac{1}{4}$	$([\mathbf{n_1}]_2), ([\overline{\mathbf{n_3}}]_2), (\mathbf{2n_0}, \mathbf{n_2}), (\overline{\mathbf{n_2}}, \mathbf{2n_4}), (\overline{\mathbf{n_1}}, \mathbf{n_3})$
$\phi_3 = -\frac{3}{8}$	$(\mathbf{2n_0}, \overline{\mathbf{n_3}}), (\mathbf{n_1}, \mathbf{2n_4}), (\overline{\mathbf{n_1}}, \overline{\mathbf{n_2}}), (\mathbf{n_2}, \mathbf{n_3})$

Table 16: This table gives the untwisted spectrum of the $\mathbb{Z}_8$-I orbifold models.

The four dimensional gauge group becomes

$$\mathrm{SO}(2n_0) \times \mathrm{U}(n_1) \times \mathrm{U}(n_2) \times \mathrm{SO}(2n_3)\,. \tag{C.2}$$

The null-solutions of the level matching condition are given by

$$3v_\nu^2 \equiv 0 \quad \Leftrightarrow \quad \nu = (12p_1, 3p_2, 4p_3) + q_1(4,-1,0) + q_2(9,0,-1)\,. \tag{C.3}$$

There are two inequivalent way a $\mathbb{Z}_6$ acts on the six-torus distinguished by the form of the spacetime shift vector: A particular solution of the level matching condition (C.1) is given by $n = (6,0,0)$.

The matter spectrum of $\mathbb{Z}_6$-I models built as follows: The untwisted matter is given in table 13. There are five twisted sector of which the first is conjugate to the fifth, and the second conjugate to the fourth. The third twisted sector obtained from the six dimensional $\mathbb{Z}_2$ sector described in table 9 in subsection 4.3.3. The first twisted sector has been collected in tables 14 and 15. Finally the second twisted sector arises as the first twisted sector of a $\mathbb{Z}_3$ orbifold with gauge shift

$$\tilde{v}_2^{\mathrm{vec}} = \frac{1}{3}\big(0^{n_0+n_3}, 1^{n_1+n_2}\big)\,, \tag{C.4}$$

in the standard $\mathbb{Z}_3$ ordering. The $\mathbb{Z}_3$ twisted matter has been collected in table 6 (or 1). To interpolate between the $\mathbb{Z}_6$ to the $\mathbb{Z}_3$ ordering of shift vectors, the matrix

$$S_2 = \left(\begin{array}{c|c|c|c} \mathbb{1}_{n_0} & & & \\ \hline & & \mathbb{1}_{n_1} & \\ \hline & & & -\mathbb{1}_{n_2} \\ \hline & \mathbb{1}_{n_3} & & \end{array}\right) \tag{C.5}$$

has been employed. Both the second and third twisted sectors are subject to appropriate identifications and further orbifolding or projections, depending to which fixed points these states are associated.

D. $\mathbb{Z}_8$ models

The $\mathbb{Z}_8$-I models have space shift $\phi = \frac{1}{8}(1,2,-3)$. Their spectra are listed in tables 16–18 on the next page. The gauge shift and the level matching condition read for the $\mathbb{Z}_8$ theory

$$v = \frac{1}{6}\big(0^{n_0}, 1^{n_1}, 2^{n_2}, 3^{n_3}, 4^{n_4}\big)\,, \qquad 4v^2 = \frac{1}{16}n_1 + \frac{1}{4}n_2 + \frac{9}{16}n_3 + n_4 \equiv \frac{7}{8}\,. \tag{D.1}$$

$\tilde{N}$	0	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{2}$	$\frac{5}{8}$
states	$\vert 0\rangle$	$\vert \underline{1}\rangle$	$\vert \underline{1}^2\rangle, \vert \underline{2}\rangle$	$\vert \underline{1}^3\rangle, \vert \underline{12}\rangle, \vert 3\rangle$	$\vert \underline{1}^4\rangle, \vert \underline{1}^2\underline{2}\rangle, \vert \underline{2}^2\rangle, \vert \underline{1}3\rangle$	$\vert \underline{1}^5\rangle, \vert \underline{1}^3\underline{2}\rangle, \vert \underline{12}^2\rangle, \vert \underline{1}^2 3\rangle, \vert \underline{2}3\rangle, \vert \underline{3}\rangle$
multi.	1	1	2	3	4	6

Table 17: The possible values of $\tilde{N}$ for the first twisted sector of a $\mathbb{Z}_8$-I are $0, \frac{1}{8}, \dots \frac{5}{8}$. The index structure of the corresponding states and total multiplicities are indicated. The holonomy group is trivially $\mathrm{U}(1)^3$.

vectorial repr. $\backslash$ $n_1+4n_2+9n_3+16n_4$	14	30	46	62	78	94
$(\mathbf{2}_+^{\mathbf{n_4-1}})$	$\frac{5}{8}$	$\frac{4}{8}$	$\frac{3}{8}$	$\frac{2}{8}$	$\frac{1}{8}$	0
$(\overline{\mathbf{n_3}}, \mathbf{2}_-^{\mathbf{n_4-1}})$	$\frac{4}{8}$	$\frac{3}{8}$	$\frac{2}{8}$	$\frac{1}{8}$	0	
$(\overline{\mathbf{n_2}}, \mathbf{2}_-^{\mathbf{n_4-1}})$, $([\overline{\mathbf{n_3}}]_2, \mathbf{2}_+^{\mathbf{n_4-1}})$	$\frac{3}{8}$	$\frac{2}{8}$	$\frac{1}{8}$	0		
$(\overline{\mathbf{n_1}}, \mathbf{2}_-^{\mathbf{n_4-1}})$, $(\overline{\mathbf{n_2}}, \overline{\mathbf{n_3}}, \mathbf{2}_+^{\mathbf{n_4-1}})$, $([\overline{\mathbf{n_3}}]_3, \mathbf{2}_-^{\mathbf{n_4-1}})$	$\frac{2}{8}$	$\frac{1}{8}$	0			
$(\mathbf{2n_0}, \mathbf{2}_-^{\mathbf{n_4-1}})$, $(\overline{\mathbf{n_1}}, \overline{\mathbf{n_3}}, \mathbf{2}_+^{\mathbf{n_4-1}})$, $([\overline{\mathbf{n_2}}]_2, \mathbf{2}_+^{\mathbf{n_4-1}})$, $(\overline{\mathbf{n_2}}, [\overline{\mathbf{n_3}}]_2, \mathbf{2}_-^{\mathbf{n_4-1}})$	$\frac{1}{8}$	0				
$(\mathbf{2n_0}, \overline{\mathbf{n_3}}, \mathbf{2}_+^{\mathbf{n_4-1}})$, $(\overline{\mathbf{n_1}}, \overline{\mathbf{n_2}}, \mathbf{2}_+^{\mathbf{n_4-1}})$, $([\overline{\mathbf{n_2}}]_2, \overline{\mathbf{n_3}}, \mathbf{2}_-^{\mathbf{n_4-1}})$, $(\mathbf{n_1}, \mathbf{2}_-^{\mathbf{n_4-1}})$	0					$\tilde{N}$

spinorial repr. $\backslash$ $n_3+4n_2+9n_1+16n_0$	14	30	46	62	78	94
$(\mathbf{2}_\alpha^{\mathbf{n_0-1}})$	$\frac{5}{8}$	$\frac{4}{8}$	$\frac{3}{8}$	$\frac{2}{8}$	$\frac{1}{8}$	0
$(\mathbf{2}_{-\alpha}^{\mathbf{n_0-1}}, \mathbf{n_1})$	$\frac{4}{8}$	$\frac{3}{8}$	$\frac{2}{8}$	$\frac{1}{8}$	0	
$(\mathbf{2}_{-\alpha}^{\mathbf{n_0-1}}, \mathbf{n_2})$, $(\mathbf{2}_\alpha^{\mathbf{n_0-1}}, [\mathbf{n_1}]_2)$	$\frac{3}{8}$	$\frac{2}{8}$	$\frac{1}{8}$	0		
$(\mathbf{2}_{-\alpha}^{\mathbf{n_0-1}}, \mathbf{n_3})$, $(\mathbf{2}_\alpha^{\mathbf{n_0-1}}, \mathbf{n_1}, \mathbf{n_2})$, $(\mathbf{2}_{-\alpha}^{\mathbf{n_0-1}}, [\mathbf{n_1}]_3)$	$\frac{2}{8}$	$\frac{1}{8}$	0			
$(\mathbf{2}_{-\alpha}^{\mathbf{n_0-1}}, \mathbf{2n_4})$, $(\mathbf{2}_\alpha^{\mathbf{n_0-1}}, \mathbf{n_1}, \mathbf{n_3})$, $(\mathbf{2}_\alpha^{\mathbf{n_0-1}}, [\mathbf{n_2}]_2)$, $(\mathbf{2}_{-\alpha}^{\mathbf{n_0-1}}, [\mathbf{n_1}]_2, \mathbf{n_2})$	$\frac{1}{8}$	0				
$(\mathbf{2}_\alpha^{\mathbf{n_0-1}}, \mathbf{n_1}, \mathbf{2n_4})$, $(\mathbf{2}_\alpha^{\mathbf{n_0-1}}, \mathbf{n_2}, \mathbf{n_3})$, $(\mathbf{2}_\alpha^{\mathbf{n_0-1}}, \mathbf{n_1}, [\mathbf{n_2}]_2)$, $(\mathbf{2}_{-\alpha}^{\mathbf{n_0-1}}, \overline{\mathbf{n_3}})$	0					$\tilde{N}$

Table 18: This table gives the first twisted sector states for the $\mathbb{Z}_8$-I model. $\alpha = (-1)^{n_0}$.

The resulting gauge group becomes

$$\mathrm{SO}(2n_0) \times \mathrm{U}(n_1) \times \mathrm{U}(n_2) \times \mathrm{U}(n_3) \times \mathrm{SO}(2n_4)\,. \tag{D.2}$$

The null-solutions of the level matching condition are given by

$$4v_\nu^2 \equiv 0 \quad \Leftrightarrow \quad \nu = (16p_1, 4p_2, 16p_3, p_4) + q_1(4,-1,0,0) + q_2(9,0,-1,0)\,. \tag{D.3}$$

A particular solution to the modular invariance condition (D.1) is given by $n = (5,0,1,0)$.

The untwisted sector is listed in table 16. The twisted sector that describe four dimensional matter with the same chirality as the untwisted states are the first, second and fifth twisted sectors. The first twisted sector states are given by tables 17 and 18. The fifth twisted sector is related to the first twisted sector by the matrix

$$S_5 = \left(\begin{array}{c|c|c|c|c} \mathbb{1}_{n_0} & & & & \\ \hline & & & -\mathbb{1}_{n_3} & \\ \hline & & \mathbb{1}_{n_2} & & \\ \hline & -\mathbb{1}_{n_1} & & & \\ \hline & & & & \mathbb{1}_{n_4} \end{array}\right) \tag{D.4}$$

via $\tilde{v}_5^{\mathrm{vec}} S_5 = v$, and in addition the spacetime indices need to be interchanged. Concretely, this means that in table 17 we interchange $1 \leftrightarrow 3$, and in table 18 we map: $\mathbf{n_1} \to \overline{\mathbf{n_3}}$

and $\mathbf{n_3} \to \overline{\mathbf{n_1}}$. The chirality of the spinorial representations are also modified due to the presence of a non-trivial d_5. In particular the $+$ chirality of the spinorial states related to vectorial weights in table 18 is replaced by $(-1)^{n_1+n_2}$, while the α chirality appearing in the spinorial states related to spinorial weights is replaced by $\alpha = (-1)^{n_0+n_2+n_3}$.

The second twisted sector is not irreducible as it is obtained from the first twisted sector of $\mathbb{Z}_4$ orbifolds. The relevant information for these spectra are given in tables 7 and 8. However, to use these tables for the second twisted sector of the $\mathbb{Z}_8$-I model one should interchange the space indices $2 \to 3$ in table 7 and use the matrix

$$S_2 = \left(\begin{array}{c|c|c|c|c} \mathbb{1}_{n_0} & & & & \\ \hline & & \mathbb{1}_{n_2} & & \\ \hline & & & & \mathbb{1}_{n_4} \\ \hline & & & -\mathbb{1}_{n_3} & \\ \hline & \mathbb{1}_{n_2} & & & \end{array}\right) \tag{D.5}$$

to bring $\tilde{v}_2^{\mathrm{vec}}$ in the standard form of a $\mathbb{Z}_4$ shift.

References

[1] D.J. Gross, J.A. Harvey, E.J. Martinec and R. Rohm, *Heterotic string theory, 1. The free heterotic string*, *Nucl. Phys.* **B 256** (1985) 253.

[2] D.J. Gross, J.A. Harvey, E.J. Martinec and R. Rohm, *Heterotic string theory, 2. The interacting heterotic string*, *Nucl. Phys.* **B 267** (1986) 75.

[3] P. Candelas, G.T. Horowitz, A. Strominger and E. Witten, *Vacuum configurations for superstrings*, *Nucl. Phys.* **B 258** (1985) 46.

[4] L.J. Dixon, J.A. Harvey, C. Vafa and E. Witten, *Strings on orbifolds*, *Nucl. Phys.* **B 261** (1985) 678.

[5] L.J. Dixon, J.A. Harvey, C. Vafa and E. Witten, *Strings on orbifolds, 2*, *Nucl. Phys.* **B 274** (1986) 285.

[6] L.J. Dixon, D. Friedan, E.J. Martinec and S.H. Shenker, *The conformal field theory of orbifolds*, *Nucl. Phys.* **B 282** (1987) 13.

[7] L.E. Ibáñez, J. Mas, H.-P. Nilles and F. Quevedo, *Heterotic strings in symmetric and asymmetric orbifold backgrounds*, *Nucl. Phys.* **B 301** (1988) 157.

[8] S. Ferrara, L. Girardello and H.P. Nilles, *Breakdown of local supersymmetry through gauge fermion condensates*, *Phys. Lett.* **B 125** (1983) 457.

[9] J.P. Derendinger, L.E. Ibáñez and H.P. Nilles, *On the low-energy $D = 4$, $N = 1$ supergravity theory extracted from the $d = 10$, $N = 1$ superstring*, *Phys. Lett.* **B 155** (1985) 65.

[10] M. Dine, R. Rohm, N. Seiberg and E. Witten, *Gluino condensation in superstring models*, *Phys. Lett.* **B 156** (1985) 55.

[11] J.A. Casas, M. Mondragon and C. Muñoz, *Reducing the number of candidates to standard model in the Z(3) orbifold*, *Phys. Lett.* **B 230** (1989) 63.

[12] Y. Katsuki, Y. Kawamura, T. Kobayashi, N. Ohtsubo and K. Tanioka, *Gauge groups of Z_n orbifold models*, *Prog. Theor. Phys.* **82** (1989) 171.

JHEP12(2004)063

[13] Y. Katsuki et al., *Z_n orbifold models*, *Nucl. Phys.* **B 341** (1990) 611.

[14] Y. Kawamura and T. Kobayashi, *Flat directions in Z_{2n} orbifold models*, *Nucl. Phys.* **B 481** (1996) 539 [hep-th/9606189].

[15] K.-S. Choi, K. Hwang and J.E. Kim, *Dynkin diagram strategy for orbifolding with Wilson lines*, *Nucl. Phys.* **B 662** (2003) 476 [hep-th/0304243].

[16] K.-S. Choi, *Spectrum of heterotic string on orbifold*, hep-th/0405195.

[17] J. Giedt, *Completion of standard model-like embeddings*, *Ann. Phys. (NY)* **289** (2001) 251 [hep-th/0009104].

[18] J. Giedt, *Spectra in standard-like Z_3 orbifold models*, *Ann. Phys. (NY)* **297** (2002) 67 [hep-th/0108244].

[19] S. Förste, H.P. Nilles, P.K.S. Vaudrevange and A. Wingerter, *Heterotic brane world*, *Phys. Rev.* **D 70** (2004) 106008 [hep-th/0406208].

[20] H.P. Nilles, *Five golden rules for superstring phenomenology*, hep-th/0410160.

[21] G. Pradisi and A. Sagnotti, *Open string orbifolds*, *Phys. Lett.* **B 216** (1989) 59.

[22] J. Polchinski, *Dirichlet-branes and Ramond-Ramond charges*, *Phys. Rev. Lett.* **75** (1995) 4724 [hep-th/9510017].

[23] E.G. Gimon and J. Polchinski, *Consistency conditions for orientifolds and D-manifolds*, *Phys. Rev.* **D 54** (1996) 1667 [hep-th/9601038].

[24] R. Blumenhagen, L. Görlich and B. Körs, *Supersymmetric 4D orientifolds of type-IIA with d6-branes at angles*, *J. High Energy Phys.* **01** (2000) 040 [hep-th/9912204].

[25] G. Pradisi, *Type i vacua from diagonal Z_3-orbifolds*, *Nucl. Phys.* **B 575** (2000) 134 [hep-th/9912218].

[26] S. Förste, G. Honecker and R. Schreyer, *Supersymmetric $Z_N \times Z_m$ orientifolds in 4d with D-branes at angles*, *Nucl. Phys.* **B 593** (2001) 127 [hep-th/0008250].

[27] S. Förste, G. Honecker and R. Schreyer, *Orientifolds with branes at angles*, *J. High Energy Phys.* **06** (2001) 004 [hep-th/0105208].

[28] R. Blumenhagen, B. Körs, D. Lüst and T. Ott, *The standard model from stable intersecting brane world orbifolds*, *Nucl. Phys.* **B 616** (2001) 3 [hep-th/0107138].

[29] M. Cvetič, G. Shiu and A.M. Uranga, *Chiral four-dimensional $N = 1$ supersymmetric type-IIA orientifolds from intersecting d6-branes*, *Nucl. Phys.* **B 615** (2001) 3 [hep-th/0107166].

[30] J. Polchinski and E. Witten, *Evidence for heterotic - type-I string duality*, *Nucl. Phys.* **B 460** (1996) 525 [hep-th/9510169].

[31] J. Giedt, *Z_3 orbifolds of the* SO(32) *heterotic string: 1 Wilson line embeddings*, *Nucl. Phys.* **B 671** (2003) 133 [hep-th/0301232].

[32] J. Giedt, *Lack of trinification in Z_3 orbifolds of the* SO(32) *heterotic string*, hep-ph/0402201.

[33] M. Dine and M. Graesser, *Cpt and other symmetries in string/M theory*, hep-th/0409209.

[34] N. Arkani-Hamed, S. Dimopoulos and G.R. Dvali, *The hierarchy problem and new dimensions at a millimeter*, *Phys. Lett.* **B 429** (1998) 263 [hep-ph/9803315].

JHEP12(2004)063

[35] I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos and G.R. Dvali, *New dimensions at a millimeter to a fermi and superstrings at a TeV*, *Phys. Lett.* **B 436** (1998) 257 [hep-ph/9804398].

[36] N. Arkani-Hamed, S. Dimopoulos and G.R. Dvali, *Phenomenology, astrophysics and cosmology of theories with sub-millimeter dimensions and TeV scale quantum gravity*, *Phys. Rev.* **D 59** (1999) 086004 [hep-ph/9807344].

[37] R. Barbieri, L.J. Hall and Y. Nomura, *A constrained standard model from a compact extra dimension*, *Phys. Rev.* **D 63** (2001) 105007 [hep-ph/0011311].

[38] A. Delgado, A. Pomarol and M. Quiros, *Electroweak and flavor physics in extensions of the standard model with large extra dimensions*, *J. High Energy Phys.* **01** (2000) 030 [hep-ph/9911252].

[39] A. Delgado, A. Pomarol and M. Quiros, *Supersymmetry and electroweak breaking from extra dimensions at the TeV-scale*, *Phys. Rev.* **D 60** (1999) 095008 [hep-ph/9812489].

[40] A. Hebecker and J. March-Russell, *The structure of GUT breaking by orbifolding*, *Nucl. Phys.* **B 625** (2002) 128 [hep-ph/0107039].

[41] L.J. Hall, Y. Nomura, T. Okui and D.R. Smith, SO(10) *unified theories in six dimensions*, *Phys. Rev.* **D 65** (2002) 035008 [hep-ph/0108071].

[42] T. Asaka, W. Buchmuller and L. Covi, *Gauge unification in six dimensions*, *Phys. Lett.* **B 523** (2001) 199 [hep-ph/0108021].

[43] A. Hebecker and M. Ratz, *Group-theoretical aspects of orbifold and conifold GUTs*, *Nucl. Phys.* **B 670** (2003) 3 [hep-ph/0306049].

[44] Y. Kawamura, *Triplet-doublet splitting, proton stability and extra dimension*, *Prog. Theor. Phys.* **105** (2001) 999 [hep-ph/0012125].

[45] I. Antoniadis, *A possible new dimension at a few TeV*, *Phys. Lett.* **B 246** (1990) 377.

[46] T. Kobayashi, S. Raby and R.-J. Zhang, *Constructing 5d orbifold grand unified theories from heterotic strings*, *Phys. Lett.* **B 593** (2004) 262 [hep-ph/0403065].

[47] T. Kobayashi, S. Raby and R.-J. Zhang, *Searching for realistic 4d string models with a Pati-Salam symmetry: orbifold grand unified theories from heterotic string compactification on a Z_6 orbifold*, *Nucl. Phys.* **B 704** (2005) 3 [hep-ph/0409098].

[48] A. Hebecker and M. Trapletti, *Gauge unification in highly anisotropic string compactifications*, hep-th/0411131.

[49] S. Groot Nibbelink, M. Hillenbach, T. Kobayashi and M.G.A. Walter, *Localization of heterotic anomalies on various hyper surfaces of T_6/Z_4*, *Phys. Rev.* **D 69** (2004) 046001 [hep-th/0308076].

[50] G. Aldazabal, A. Font, L.E. Ibáñez, A.M. Uranga and G. Violero, *Non-perturbative heterotic $D = 6, 4$, $N = 1$ orbifold vacua*, *Nucl. Phys.* **B 519** (1998) 239 [hep-th/9706158].

[51] J. Polchinski, *String theory, 2. Superstring theory and beyond*, Cambridge University Press, Cambridge 1998.

[52] F. Gmeiner, S. Groot Nibbelink, H.P. Nilles, M. Olechowski and M.G.A. Walter, *Localized anomalies in heterotic orbifolds*, *Nucl. Phys.* **B 648** (2003) 35 [hep-th/0208146].

[53] R. Slansky, *Group theory for unified model building*, *Phys. Rept.* **79** (1981) 1.

[54] E. Witten, *Toroidal compactification without vector structure*, *J. High Energy Phys.* **02** (1998) 006 [hep-th/9712028].

[55] J. de Boer et al., *Triples, fluxes and strings*, *Adv. Theor. Math. Phys.* **4** (2002) 995 [hep-th/0103170].

[56] G. Aldazabal, D. Badagnani, L.E. Ibáñez and A.M. Uranga, *Tadpole versus anomaly cancellation in $D = 4, 6$ compact IIB orientifolds*, *J. High Energy Phys.* **06** (1999) 031 [hep-th/9904071].

[57] C. Angelantonj, M. Bianchi, G. Pradisi, A. Sagnotti and Y.S. Stanev, *Chiral asymmetry in four-dimensional open-string vacua*, *Phys. Lett.* **B 385** (1996) 96 [hep-th/9606169].

[58] G. Aldazabal, A. Font, L.E. Ibáñez and G. Violero, *$D = 4$, $N = 1$, type-IIB orientifolds*, *Nucl. Phys.* **B 536** (1998) 29 [hep-th/9804026].

[59] Z. Lalak, S. Lavignac and H.P. Nilles, *String dualities in the presence of anomalous* U(1) *symmetries*, *Nucl. Phys.* **B 559** (1999) 48 [hep-th/9903160].

[60] Z. Lalak, S. Lavignac and H.P. Nilles, *Target-space duality in heterotic and type-I effective lagrangians*, *Nucl. Phys.* **B 576** (2000) 399 [hep-th/9912206].

[61] J. Polchinski, S. Chaudhuri and C.V. Johnson, *Notes on D-branes*, hep-th/9602052.

[62] Z. Kakushadze, *Aspects of $N = 1$ type-I-heterotic duality in four dimensions*, *Nucl. Phys.* **B 512** (1998) 221 [hep-th/9704059].

[63] Z. Kakushadze, G. Shiu and S.H.H. Tye, *Type iib orientifolds, F-theory, type-I strings on orbifolds and type-I heterotic duality*, *Nucl. Phys.* **B 533** (1998) 25 [hep-th/9804092].

[64] L.E. Ibáñez, R. Rabadán and A.M. Uranga, *Anomalous* U(1)*'s in type-I and type-IIB $D = 4$, $N = 1$ string vacua*, *Nucl. Phys.* **B 542** (1999) 112 [hep-th/9808139].

[65] J.F. Morales, C.A. Scrucca and M. Serone, *Anomalous couplings for D-branes and O-planes*, *Nucl. Phys.* **B 552** (1999) 291 [hep-th/9812071].

[66] C.A. Scrucca and M. Serone, *Gauge and gravitational anomalies in $D = 4$ $N = 1$ orientifolds*, *J. High Energy Phys.* **12** (1999) 024 [hep-th/9912108].

[67] A. Dabholkar and J. Park, *A note on orientifolds and F-theory*, *Phys. Lett.* **B 394** (1997) 302 [hep-th/9607041].

[68] J.D. Blum and A. Zaffaroni, *An orientifold from F-theory*, *Phys. Lett.* **B 387** (1996) 71 [hep-th/9607019].

[69] M. Klein and R. Rabadán, *$D = 4$, $N = 1$ orientifolds with vector structure*, *Nucl. Phys.* **B 596** (2001) 197 [hep-th/0007087].

[70] M. Klein and R. Rabadán, *$Z_N \times Z_M$ orientifolds with and without discrete torsion*, *J. High Energy Phys.* **10** (2000) 049 [hep-th/0008173].

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